

N° d'ordre : A.O.10.124

T H E S E

présentée

A L'UNIVERSITE DE PROVENCE

pour obtenir

LE GRADE DE DOCTEUR ES SCIENCES PHYSIQUES

par

Jean BELLISSARD

CHAMPS QUANTIFIES DANS UN

CHAMP EXTERIEUR

Soutenue, le 28 Juin 1974, devant la Commission d'Examen

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Jury		D. KASTLER	} Examineurs
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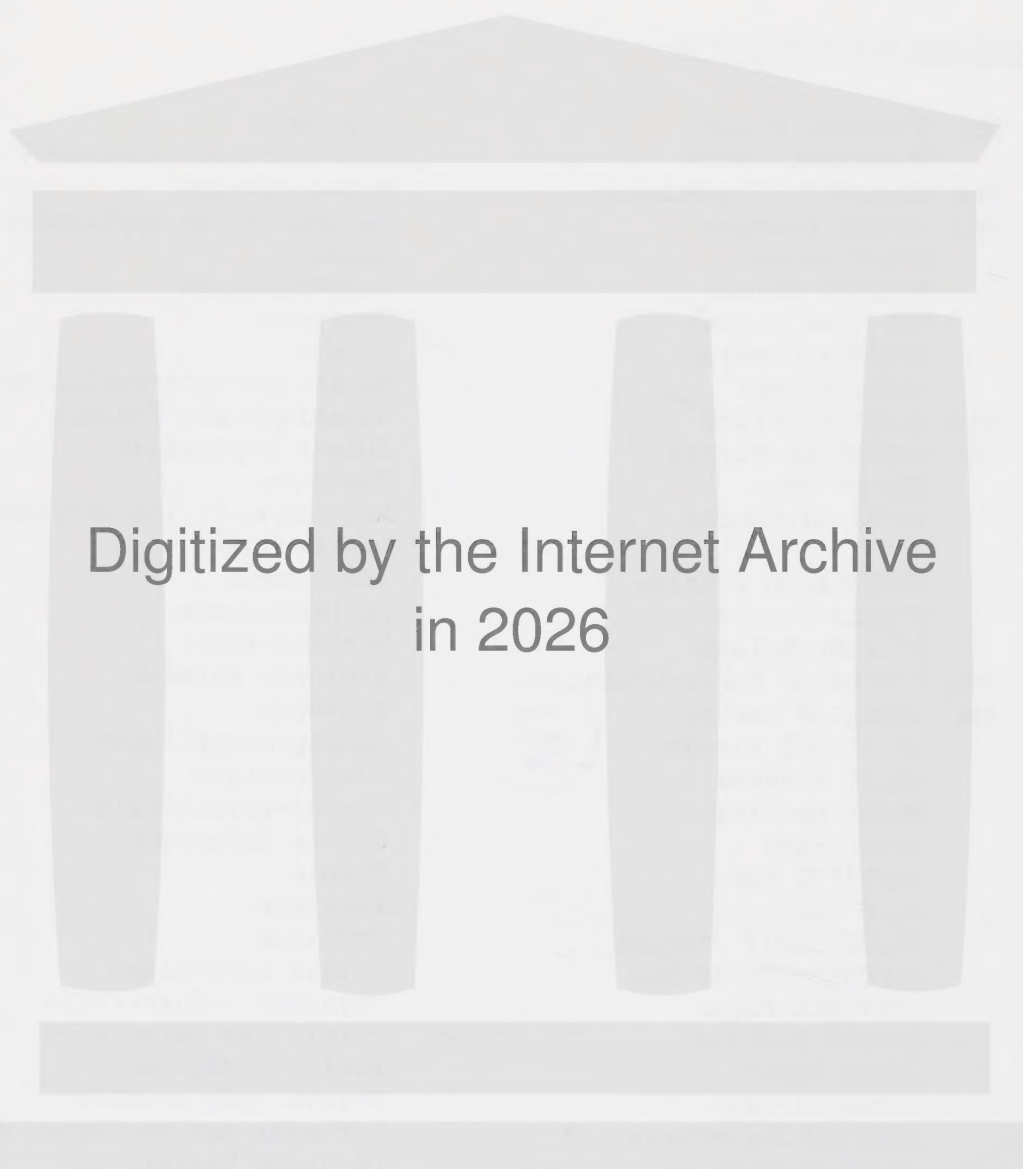


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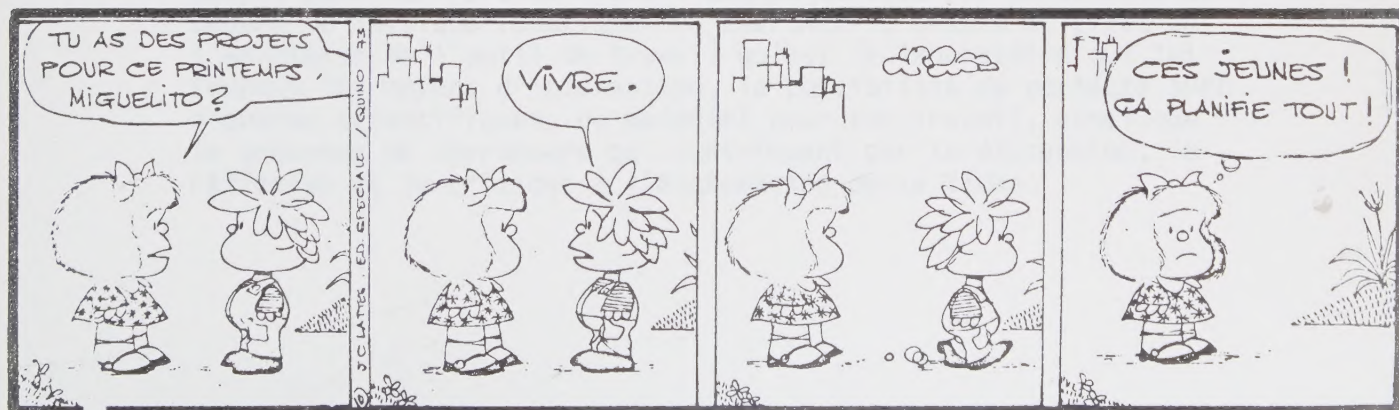
aux Cinq Meilleurs

à Salvador

à Loulou

à vos souhaits

MERCI.



- R E M E R C I E M E N T S -

La coutume veut qu'un postulant à une thèse remercie tous ceux qui l'ont aidé dans sa tâche. Bien qu'il ne s'agisse pas de la partie la plus aisée, c'est très volontiers que je m'exécuterai.

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Et chaque jour m'apporte la conviction que la distinction faite entre bons et mauvais qu'ils soient techniciens, administratifs ou chercheurs, n'a pour but que de pouvoir exploiter à bon marché une force de travail indispensable au fonctionnement d'un laboratoire. Il s'agit là d'un des multiples moyens employés par une classe minoritaire pour utiliser la recherche comme source de profits sans se préoccuper d'améliorer la condition de vie de la population.

C'est pourquoi les remerciements que j'adresse à tous mes camarades du laboratoire, et en particulier à Maryse Cohen-Solal et à Nicole Jean qui se sont dépensées sans compter pour la réalisation de ce travail, ne sont ni purement académiques, ni seulement sentimentaux, mais liés à la conviction profonde que leur rôle est fondamental, et je souhaite qu'ils en récoltent aussi les fruits.

ADDENDUM

Le Président du jury tient à préciser que certains membres du jury ne se sentent pas liés par des opinions exprimées en page i et ii (remerciements) de cette thèse , opinions qu'ils n'entendent en aucune façon cautionner .



INTRODUCTION : CHAMPS QUANTIFIES DANS UN CHAMP EXTERIEUR

Un des problèmes les plus simples de la théorie quantique des champs consiste à étudier un champ quantique relativiste couplé à un champ extérieur classique par un Lagrangien d'interaction bilinéaire dans le champ quantifié.

La théorie perturbative correspondante est traitée dans un certain nombre d'articles et d'ouvrages de théorie des champs. Le problème se pose cependant de savoir si les développements perturbatifs peuvent être resumés de façon à définir une théorie quantique d'opérateurs dans un espace de Hilbert. Ce problème a été soulevé en 1968 par A.S. Wightman et ses collaborateurs, et a reçu un certain nombre de réponses partielles.

Cette thèse apporte à ces problèmes de nouveaux éléments de solutions. Le travail présenté ici comporte trois parties qui ont fait chacune l'objet d'un article prêt à être publié.

La première partie passe en revue la construction des séries perturbatives et décrit certains de leurs aspects du point de vue de l'analyse fonctionnelle. De cette théorie, on extrait des propriétés qui, érigées en axiomes, définissent les théories non perturbatives correspondantes. Le résultat essentiel est que si la fonction de Green à deux points possède certaines propriétés il existe un opérateur de champ de Heisenberg et un opérateur de diffusion qui résolvent le problème posé, dans l'espace de Fock du champ libre entrant.

La deuxième partie est consacrée à la démonstration de théorèmes d'existence qui assurent que les axiomes précédemment posés, sont satisfaits pour des classes assez générales de champs extérieurs. On y démontre également dans certains cas la convergence des séries perturbatives pour les fonctions de Green et l'opérateur de diffusion.

Enfin une dernière partie est consacrée à la classification des champs libres, irréductibles, covariants, relativistes et locaux. On y trouvera une forme compacte et maniable de résultats énoncés par J.J. Loeffel, J.P. Laederman, et P. Renouard, qui sont utilisés dans la seconde partie sous la forme présentée ici.

QUANTIZED FIELDS IN INTERACTION WITH EXTERNAL FIELDS

J. BELLISSARD (*)

I- EXACT SOLUTIONS AND PERTURBATIVE EXPANSIONS

Abstract : We consider a massive, charged, scalar quantized field interacting with an external classical field. Guided by renormalized perturbation theory we show that whenever the integral equations defining the Feynman or retarded or advanced interaction kernel possess non perturbative solutions, there exists a S operator which satisfies, up to a phase, the axioms of Bogoliubov, and is given for small external fields by a power series which converges on coherent states. Furthermore this construction is shown to be equivalent to the one based on the Yang-Feldman equation. This is a consequence of the relations between chronological and retarded Green's functions which are described in detail.

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Introduction

Numerous papers have been devoted to the study of the interactions of particles with external fields, within the framework of quantum field theory.

The formal aspects were well developed twenty years ago, in particular through the work of Feynman [1], Matthews and Salam [2] and Schwinger [3]. A good summary can be found e.g. in Thirring's book on Quantum Electrodynamics [4].

Mathematically rigorous non perturbative treatments were given in some particular cases by several authors. Capri [5] has explained lucidly "the reduction to c-number problem" (cf. also Wightman [6]). Bongaarts [7] has treated the case of spin $1/2$ particles in a stationary external field. Seiler [8], using the results of a paper by Schroer, Seiler and Swieca [9], proved the existence of a fixed time evolution operator in the following cases: scalar and pure electric external fields for spin 0 and spin $1/2$ fields. Wightman [6] has given necessary and sufficient conditions for the existence of a unitary S-operator for arbitrary spins, in the case of external field coupled with quantized fields which fulfil a first order system of partial differential equations.

The perturbative aspects of quantum field theory described for instance in Bogoliubov and Shirkov's classical book [10] have recently been further developed by Epstein and Glaser [11], Steinmann [12] and Zimmermann [13].

These authors have proved the existence of an S-operator as an operator valued formal power series, for arbitrary Wick polynomial Lagrangeans.

The aim of this paper is to exhibit the connection between Bogoliubov's version of perturbation theory and non perturbative methods developed by the above mentioned authors. For the sake of simplicity

we shall consider the problem of a charged scalar boson field interacting with a classical external field. Possible extensions to more general cases will be mentioned at each step. The main problem is to construct a solution of the Bogoliubov axioms as an operator in the Fock space of the in-field.

In section I, we first consider $\mathcal{S}$ as a formal power series with respect to the external field. We show (theorem I.1.1, section I.2 and proposition I.3.3) that $\mathcal{S}$ is, up to a phase, equal to explicit function $\mathcal{S}_0(I)$ of the two-point interaction kernel "I". This is, in the present framework, "the reduction to c-number problem". It can be seen that "I" is defined, in perturbation theory by an integral equation (cf. equation II.0.1)

It seems therefore reasonable, in the perspective of a non perturbative treatment, to find exact solutions of this equation : we then define a class of two-point functions, which we call N-P kernels (section II.2), which will enable us to construct $\mathcal{S}_0(I)$, and we study some properties of N-P solutions of the defining equation for "I".

In particular we explicitly establish the connection between methods using the hamiltonian formalism [8] the Yang-Källén -Feldman equation [5, 6, 14], and the perturbative theory [1, 2, 3, 4] (cf. theorem II.4.1, and section II.5). As a corollary, the two-points function "I" can be computed by solving the Cauchy problem for the Klein-Gordon equation in an external field (theorem II.4.1, remark II.2.11).

In the case of scalar boson fields, we also show the redundancy of some of the necessary and sufficient conditions given by Wightman [6] for the existence of the S-operator (cf. prop. II.5.2).

In the last section, assuming the existence of an N-P interaction kernel "I", we show that $\mathcal{S}_o(I)$ defines in Fock space, a covariant, unitary and causal up to a phase, S-operator in the sense of Bogoliubov [10]. We also prove some results about analyticity of $\mathcal{S}_o(I)$ with respect to "I". We construct an interpolating field, which is solution of the Yang-Källén-Feldman equations for which the in and out field are related by the S-operator $\mathcal{S}_o(I)$.

This construction explicitly made for a charged scalar field, can be extended to fields with arbitrary spins if one is careful enough. At each step, we indicate how this can be done.

It seems that some difficulties appear in the case of fermion fields although it is possible to define an S-operator (cf. Labonte [15]).

In a forth coming paper [16] , we will show the existence of N-P solutions, in the cases of spin 0 1/2 and 1 fields, for special choices of the external field.

I - PERTURBATIVE THEORY

I.1-The Bogoliubov S-operator.

The S-operator, which describes the interaction of a quantum field with an external field, is easily constructed in the framework of Bogoliubov's perturbation theory [10], by applying the Epstein-Glaser formalism [11].

In the simple case of a charged scalar boson, interacting with a scalar field and with an electromagnetic vector potential the corresponding (quadratic) lagrangean density is :

$$\mathcal{L}_I(x; g) = v(x) : \varphi^* \varphi(x) + i A_\mu(x) : \varphi^* \overleftrightarrow{\partial}^\mu \varphi(x) \quad (\text{I.1.1})$$

where $g = (v, A_0, A_1, A_2, A_3) \in \mathcal{S}(\mathbb{R}^4) \times \mathcal{S}$

and φ is the free field describing the scalar bosons (see Appendix 1). Unitarity holds if v and A_μ are real valued

By applying Wick's theorems [11, 17] (see Appendix 2) one can show the following theorem (whose proof is given in Appendix 3).

Theorem I.1.1 :

Given the lagrangean density I.1.1, there exists in the sense of formal power series a solution of the causal Bogoliubov axioms such that :

$$S(g) = (\Omega, S(g)\Omega) : \exp i \int \varphi^*(x) I(x, y) \varphi(y) dx dy : \quad (\text{I.1.2})$$

Here, "I" is the distribution kernel valued formal power series defined by :

$$I = A + A \Delta_F I = A + I \Delta_F A \quad (\text{I.1.3})$$

with :

$$A(x, y) = \delta(x-y)(v(x) + A_\mu A^\mu(x)) + \partial_\mu [\bar{A}_\mu(x) + A_\mu(y)] \partial^\mu \delta(x-y) \quad (\text{I.1.4})$$

and Δ_F the two-point Green function :

$$\Delta_F(x) = (2\pi)^{-4} \int e^{i p \cdot x} (p^2 - m^2 + i0)^{-1} d^4 p \quad (\text{I.1.5})$$

Remark I.1.2 :

Let ψ be a free (fermion or boson) field, whose two-point Green function is given by :

$$S_F(x) = \int e^{i p \cdot x} P(p) (p^2 - m^2 + i0)^{-1} d^4 p \quad (\text{I.1.6})$$

where P is a matrix valued polynomial. Let $\mathcal{L}_\psi(x; \theta)$ be the following (quadratic) lagrangean density :

$$\mathcal{L}_\psi(x; \theta) = : \bar{\psi}_\alpha(x) \psi_\beta(x) : \mathcal{O}_{\alpha\beta}(x) \quad (\text{I.1.7})$$

where $\mathcal{O}_{\alpha\beta}(x) \in \mathcal{S}(\mathbb{R}^4)$

The previous theorem holds if we replace φ by ψ , Δ_F by S_F , $\mathcal{L}_I$ by $\mathcal{L}_\psi$ and $A(x, y)$ by $\theta(x) \delta(x-y)$

Remark I.1.3 :

One can express "I" in term of Feynman graphs (see fig.1)

Remark I.1.4 :

One can see that :

$$\int d^4 x d^4 y : \varphi^*(x) \varphi(y) : A(x, y) = \int d^4 x : \mathcal{L}^{\text{can}}(x, g) \quad \text{a) } (\text{I.1.8})$$

where $\mathcal{L}^{\text{can}}$ is the canonical lagrangean for this problem :

$$:\mathcal{L}^{\text{can}}(x;g): = [v(x) + A_\mu A^\mu(x)] : \varphi^* \varphi : (x) + i \int A_\mu(x) : \varphi^* \overleftrightarrow{\partial}^\mu \varphi : (x) \quad \text{b)}$$

We note that $\mathcal{L}^{\text{can}}$ is different from $\mathcal{L}_I$ because, by definition [11, 17], the latter has to be of the first order in $(v, A_\mu) = g$. $\mathcal{L}^{\text{can}}$ can be recovered upon making the following choice of renormalization :

$$(\Omega | T \partial^\mu \varphi(x) \partial^\nu \varphi(y) \Omega) = -i [\partial^\mu \partial^\nu \Delta_F(x-y) + g^{\mu\nu} \delta(x-y)] \quad (\text{I.1.9})$$

I.2 - THE TWO POINT GREEN FUNCTIONS

The two point Green function can be defined in the sense of formal power series by the following equation⁽³⁾

$$G_F = \Delta_F + \Delta_F A G_F = \Delta_F + G_F A \Delta_F \quad (\text{I.2.1})$$

Its expression in terms of Feynman graphs is given in fig. 2. It is easy to see that G_F is equal to

$$G_F = \Delta_F + \Delta_F I \Delta_F \quad (\text{I.2.2})$$

or equivalently :

$$I = A + A G_F A \quad (\text{I.2.3})$$

Equation (I.2.1) shows us that G_F satisfies the following Klein Gordon equation

$$(\partial_{\mu,x} - i A_\mu(x) [\partial_x^\mu - i A^\mu(x)] + m^2 - v(x)) G_F(x,y) = \delta(x-y)$$

$$(\partial_{\mu,y} + i A_\mu(y) [\partial_y^\mu + i A^\mu(y)] + m^2 - v(y)) G_F(x,y) = \delta(x-y) \quad (\text{I.2.4})$$

with the conditions [3] :

if $x \notin (\text{supp } g + \bar{V}^+)$, $G_F(x, y)$ propagates only positive frequencies

if $x \notin (\text{supp } g + \bar{V}^-)$, $G_F(x, y)$ propagates only negative frequencies

I.3. THE PHYSICAL MEANING OF I :

Let $\mathcal{H}_\epsilon$ be the one particle Hilbert space with energy of the sign of $\epsilon = \pm 1$ (cf. Appendix). Let us define

$$\tilde{I}(p, q) = 2\pi^{-4} \int e^{i(p \cdot x - q \cdot y)} I(x, y) dx^4 dy^4 \quad (\text{I.3.1})$$

Let $I_{\epsilon\epsilon'}$ be the following operator from $\mathcal{H}_{\epsilon'}$ to $\mathcal{H}_\epsilon$ (cf. Appendix 1) :

$$(I_{\epsilon\epsilon'} f)(p) = 2\pi \int_{\vec{q}^2 = (\vec{q}'^2 + m^2)^{1/2}} \tilde{I}(\epsilon p, \epsilon' q') f(q') \frac{1}{2} (\vec{q}'^2 + m^2)^{-1/2} d^3 \vec{q}' \quad (\text{I.3.2})$$

with $\epsilon = \pm 1$, $\epsilon' = \pm 1$ and $f \in \mathcal{H}_{\epsilon'}$.

Then "I" defines the following operator on $\mathcal{H}_+ \oplus \mathcal{H}_-$

$$I_S = \begin{bmatrix} I_{++} & I_{+-} \\ I_{-+} & I_{--} \end{bmatrix} \quad (\text{I.3.3})$$

Proposition I.3.1 :

To any order in the external fields I_{++} , I_{--} are bounded operators, I_{+-} and I_{-+} are in the Hilbert-Schmidt class [18,19,20] .

Proof : To order "n", "I" reads :

$$\tilde{I}_n(p, q) = A \overbrace{\Delta_F \dots \Delta_F}^{n\text{-times } A} A (p, q) \quad a)$$

From Appendix 5, lemma A.5.2 and corollary A.5.6 we can deduce that $\tilde{I}_n \in \mathcal{N}(m, \alpha, 1/4)$ $\forall m \geq 0$ and $\forall \alpha \geq 0$. Using lemma A.5.3,

$$\left. \text{Tr}(I_+ I_+^*) \right|_{\text{order "n"}} = \sum_{p=0}^n \text{Tr}((I_+)_p (I_+^*)_{n-p})$$

Proposition I.3.1 then follows.

Remark I.3.2 : The results of Appendix 5 (in particular Lemma A.5.5) allow us to extend Prop. I.3.1 to the case of fields with higher spins described in Remark I.1.2 :

- I_{+-} and I_{-+} are in the Hilbert Schmidt class.
- $\tilde{I}_{++}$ (resp $\tilde{I}_{--}$) is densely defined on a domain $\mathcal{D}_+$ (resp $\mathcal{D}_-$) such that $I_{++} \mathcal{D}_+ \subset \mathcal{D}_+$ (resp $I_{--} \mathcal{D}_- \subset \mathcal{D}_-$)

To see this we have to remark that (see Remark I.1.2)

$$\forall m \geq 0 \quad \forall \alpha \geq 0 \quad \tilde{\mathcal{O}}(p, q) = \tilde{\mathcal{O}}(p-q) \in \mathcal{N}(m, \alpha, 0) \quad \text{if } \mathcal{O}(x) \in \mathcal{S}'(\mathbb{R}^4)$$

and we have to use lemmas A.5.5 and A.5.6.

Proposition I.3.3

In the sense of formal power series, the operator I_S is the building block of the following physical quantities

$$a) \quad |(\Omega, S \Omega)|^2 = \det(\mathbb{1}_{++} - I_+ (I_+)^*) \quad (I.3.6)$$

b) The probability $P_{e^+e^-}$ of creation of one pair is given by

$$P_{e^+e^-} = \det(\mathbb{1}_{++} - I_+ I_+^*) \text{Tr}(I_+ I_+^*) \quad (I.3.7)$$

c) The scattering amplitude for the process particle $\rightarrow$ particle is given by

$$(\Omega, S, \Omega) (\mathbb{1} + i I_{++}) (p, q) \quad (\text{I.3.8})$$

d) The complete S operator is given by :

$$S = e^{i\omega} S_0(I) = e^{i\omega} \det(\mathbb{1} - I_+ I_+^*)^{\frac{1}{2}} e^{i(p^+ I_+ b^+ + a^+ I_+ a^- + b^- I_- b^+ + b^- I_- a^-)} \quad (\text{I.3.9})$$

where ω is a phase depending on I .

The results of this proposition are well-known [3,4,6,8]. They can be proved by using the unitarity of the S -operator and the definitions of the out-field and the out-vacuum

$$\Omega_{out} = S^{-1} \Omega = (\Omega, S, \Omega)^{-1} e^{-i a^+ I_+^* b^+} \Omega \quad (\text{I.3.10})$$

$$\varphi_{out}(x) = S^{-1} \varphi(x) S \quad (\text{I.3.11})$$

Remark I.3.4 :

Using Nelson's notations [21] for second quantized operators one has :

$$:\exp i(a^+ I_+ a^- + b^- I_- b^+): = \Gamma((\mathbb{1} + i I_+) \otimes (\mathbb{1} + i I_-^t)) \quad (\text{I.3.12})$$

with

$$\Gamma(A \otimes B) \Phi_{r,s} = (A^{\otimes r} \otimes B^{\otimes s}) \Phi_{r,s} \quad (\text{I.3.13})$$

if $\Phi_{r,s} \in \mathcal{H}_+^{\vee r} \otimes \mathcal{H}_-^{\vee s}$ (cf Appendix 1)

Remark I.3.5

In a non perturbative framework we shall use the results of proposition I.3.3 to reduce the problem to the proof of the existence of a kernel I such that I_+, I_- are

bounded operators, and I_{+-} I_{-+} are in the Hilbert-Schmidt class.

I.4. THE INTERPOLATING FIELD

In perturbation theory the interpolating field is given by the formula [11]

$$\psi(x) = S^{-1}(g, J) \frac{\delta S}{\delta g J(x)}(g, J) \Big|_{J=0} \quad (I.4.1)$$

Here $S(g, J)$ is the Bogoliubov S -operator constructed from the lagrangean density

$$\mathcal{L}'(x; g, J) = \mathcal{L}_I(x; g) + J(x) \varphi(x) \quad (I.4.2)$$

Proposition I.4.1

The interpolating field is given by the formula :

$$\psi(x) = |S|^{-2} : e^{i \int \varphi^* \bar{I} \varphi} : \int (\delta(x-y) + \Delta_F I(x, y)) \varphi(y) dy e^{i \int \varphi^* I \varphi} : \quad (I.4.3)$$

It satisfies the Yang-Feldman-Källén equation ; [5, 14, 22]

$$\psi(x) = \varphi(x) + \int \Delta_r(x-z) A(z, y) \psi(y) dy dz \quad a)$$

(I.4.4)

$$\psi(x) = \varphi_{out}(x) + \int \Delta_a(x-z) A(z, y) \psi(y) dy dz \quad b)$$

The proof of equation (I.4.3) is given in Appendix 3. Let us give the proof of (I.4.4a)

$$\Delta_F = \Delta_r - \Delta_+ \implies (1 - \Delta_r A)(1 + \Delta_F I) = (1 - \Delta_+ I) \quad (I.4.5) \quad [3]$$

because of (I.1.3) for I .

Thus :

$$(1 - \Delta_r A) \psi = : e^{-i \int \varphi^* \bar{I} \varphi} : (1 - \Delta_+ I) \varphi e^{i \int \varphi^* I \varphi} : | < S > |^2 \quad (\text{I.4.6})$$

where

$$\bar{I}(x, y) = I(y, x)^* \quad (\text{I.4.7})$$

Wick's theorem yields :

$$:(1 - \Delta_+ I) \varphi e^{i \int \varphi^* I \varphi} : = : e^{i \int \varphi^* I \varphi} : \varphi \quad (\text{I.4.8})$$

The unitarity of $\mathcal{S}$ yields the required result. The proof of (I.4.4b) is similar.

Remark I.4.2

Capri [5] and Wightman [6] have used these equations to construct the interpolating field and the out-field for this problem. Following another approach, Seiler, Schroer and Swieca [8, 9] (see also [6]) solve these equations for c-numbers function, and prove the existence of the classical S -operator, which is in our formalism (in the sense of formal power series)

$$\mathcal{S}_{cl} = (1 - \Delta_a A) (1 - \Delta_r A)^{-1} \quad (\text{I.4.9})$$

In Capri's notations [5, 6] $\mathcal{S}_{cl} = \mathcal{T}_a \mathcal{T}_r^{-1}$ with $\mathcal{T}_a = (1 - \Delta_a A)$

Thus,

$$\varphi_{out} = \mathcal{S}_{cl} \varphi \quad (\text{I.4.10})$$

II- ALGEBRAIC PROPERTIES OF THE TWO-POINT GREEN FUNCTIONS

II.0 INTRODUCTION

In this chapter we shall investigate the algebraic properties of the solutions of the defining equations for I (I.1.3).

We shall similarly define three other kernels $\bar{I}$, J_r , J_a ,

$$\begin{aligned} I &= A + A \Delta_F I = A + I \Delta_F A & \bar{I} &= A + A \Delta_{\bar{F}} \bar{I} = A + \bar{I} \Delta_{\bar{F}} A \\ J_r &= A + A \Delta_r J_r = A + J_r \Delta_r A & J_a &= A + A \Delta_a J_a = A + J_a \Delta_a A \end{aligned} \quad (\text{II.0.1})$$

Keeping in mind how the quantum theory relies on the c-number theory and in view of the results of perturbation theory (remark I.3.5) we shall need solutions of II.0.1 with specific properties described in definitions II.2.1 and II.2.2. These solutions will be called N-P kernels (N-P = non perturbative).

We shall also define W-N-P kernels (W = weak) (cf. def. II.2.3) and will establish later [16] that for any external field in $\mathcal{C}_0^\infty(\mathbb{R}^4)$ [23] II.0.1 has one and only one W-N-P solution. N-P solution however can be shown to exist for some restricted classes of external fields [8, 9, 16]

This section is devoted to the proof of some exact relations between N-P (or W-N-P) solution of II.0.1 based on elementary properties of the free two-point functions.

Some of these relations can be found for instance in J. Schwinger's work [3], some others in A. Wightman's [6] and R. Seiler [8] more recent papers (cf. Remark I.4.2) and are related

to general properties of the Bogoliubov transformations [24] .

The main result of this section is the equivalence between the equations defining $\mathbb{I}$ and $\mathbb{J}_r$, in so far as the existence and unicity of a solution are concerned (theorem II.4.1). As a corollary (in some sense) it is necessary and sufficient, in order to solve II.0.1a) to solve the Cauchy problem for the Klein-Gordon equation in an external field [6] .

Another result is that necessary and sufficient conditions given by A. Wightman ([6] formula 4.12) for the uniqueness of the out-vacuum, is automatically verified for bosons of spin zero and real external fields (prop. II.5.2)

Everything so far mentioned can be extended without modification for fields with integer spins. For half-integer spin field there are some changes to be made which are stated at each step.

II.1 PRELIMINARIES

We first need some definitions.

Let K, L be kernels in $\mathcal{S}'(\mathbb{R}^4 \times \mathbb{R}^4)$. We define

$$\text{The Fourier transform : } \tilde{K}(p, q) = (2\pi)^{-4} \int e^{i(p \cdot x - q \cdot y)} K(x, y) dx^4 dy^4 \quad (\text{II.1.1})$$

$$\begin{aligned} &\text{The product :} \\ &(\text{if it exists}) \quad KL(x, y) = \int K(x, z) L(z, y) dz^4 \end{aligned} \quad (\text{II.1.2})$$

$$\text{The adjoint} \quad K^\dagger(x, y) = K(y, x)^* \quad (\text{II.1.3})$$

The unit kernel $\mathcal{A}(x,y) = \delta(x-y)$ (II.1.4)

The kernel associated with the convolution through a tempered distribution :

$$\mathcal{T}(x,y) = \mathcal{T}(x-y) \quad (\text{II.1.5})$$

Then :

$$\tilde{K} = K \quad \tilde{K} + \tilde{L} = \widetilde{K+L} \quad \tilde{K} \cdot \tilde{L} = \widetilde{K \cdot L} \quad \text{a)}$$

$$K^{\dagger\dagger} = K \quad K^{\dagger} + L^{\dagger} = (K+L)^{\dagger} \quad (K \cdot L)^{\dagger} = L^{\dagger} K^{\dagger} \quad \text{b)} \quad (\text{II.1.6})$$

$$(\widetilde{K^{\dagger}})^{\dagger} = (\tilde{K})^{\dagger} \quad K \cdot 1 = 1 \cdot K = K \quad \text{c)}$$

$$\tilde{1} = 1 = 1^{\dagger} \quad \text{d)}$$

The operators $\hat{K}_{\epsilon\epsilon'}$ ($\epsilon = \pm 1, \epsilon' = \pm 1$) are defined (if these expressions have a meaning) by :

$$(\hat{K}_{\epsilon\epsilon'}, f)(\vec{p}) = 2\pi \int_{q^0 = \omega_q, p^0 = \omega_p} \tilde{K}(\vec{p}, \epsilon'; q) f(\vec{q}) \frac{d^3 \vec{q}}{2\omega_q} \quad (\text{II.1.7})$$

with
and $\omega_q = (1 + \vec{q}^2)^{1/2}$

$$\hat{K} = \begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix} \quad (\text{II.1.8})$$

Recall that

$$\Delta_+(\xi) = -i(2\pi)^{-3} \int e^{i(\omega_p \xi^0 - \vec{p} \cdot \vec{\xi})} d^3\vec{p} / 2\omega_p = \Delta_-(-\xi) \quad (\text{II.1.9})$$

Then we have (in kernel notations)

$$\Delta = \Delta_+ - \Delta_- = \Delta_r - \Delta_a \quad \text{supp } \Delta_a \subset \{(x-y) \in \overline{V}^{\pm}\} \quad \text{a)}$$

$$\Delta_F = \Delta_r - \Delta_+ = \Delta_a - \Delta_- \quad \text{b) (II.1.10)}$$

$$\Delta_{\overline{F}} = \Delta_r + \Delta_- = \Delta_a + \Delta_+ \quad \text{c)}$$

$$\Delta_r^\dagger = \Delta_a \quad \Delta_\pm^\dagger = -\Delta_\pm \quad \Delta_F^\dagger = \Delta_{\overline{F}} \quad (\text{II.1.11})$$

We will also define (see equations II.0.1)

$$\mathbb{I}_S = \hat{\mathbb{I}} \quad \overline{\mathbb{I}}_S = \hat{\overline{\mathbb{I}}} \quad \mathbb{J}_R = \hat{\mathbb{J}}_r \Gamma \quad \mathbb{J}_A = \hat{\mathbb{J}}_a \Gamma \quad \text{a)}$$

$$\text{where} \quad \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{b) (II.1.12)}$$

II.2 NON PERTURBATIVE SOLUTIONS (N-P and W-N-P kernels)

Definition II.2.1 : A kernel $K \in \mathcal{S}'(\mathbb{R}^4 \times \mathbb{R}^4)$ will be called
 regular if it defines a linear continuous map from $\mathcal{O}_M(\mathbb{R}^4)$
 to $\mathcal{S}(\mathbb{R}^4)$ [23, 25].

Definition II.2.2 : A regular kernel will be called W-N-P if

- i) $\hat{K}$ is a bounded operator in $\mathcal{H}_+ \oplus \mathcal{H}_-$
- ii) $\hat{K}_+$ and $\hat{K}_-$ are compact [18, 19, 20]

Definition II.2.3 : A W- N-P kernel will be called N P if $\hat{K}_+$ and
 $\hat{K}_-$ are in the Hilbert -Schmidt class [18, 19, 20].

Proposition II.2.4 : The kernel A defined in I.1.4 is N-P

Proof : cf Appendix 5 Lemma A.5.2 and A.5.3

Proposition II.2.5 : Let $\mathcal{D}$ be the dense domain in $\mathcal{H}_+ \oplus \mathcal{H}_-$
 whose elements are functions of $\mathcal{S}(\mathbb{R}^3)$. Then if K is
 a regular kernel, $\hat{K}$ is defined on $\mathcal{D}$ and

$$\hat{K} \mathcal{D} \subset \mathcal{D} \quad (\text{II.2.1})$$

Proof : Let $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$ be a, element of $\mathcal{D}$. Then if

$$\hat{f} = \delta\left(\frac{p^0 - \omega_p}{2\omega_p}\right) f_+(\vec{p}) + \delta\left(\frac{p^0 + \omega_p}{2\omega_p}\right) f_-(\vec{p}) \quad (\text{II.2.2})$$

One can see that $\hat{f} \in \mathcal{O}'_c(\mathbb{R}^4)$ and its Fourier transform $\tilde{f}$ is in $\mathcal{O}'_M(\mathbb{R}^4)$ [25]. Therefore, the regularity of K gives :

$$K\tilde{f} \in \mathcal{S}(\mathbb{R}^4)$$

The definition of $\widehat{K}$ (cf. equation II.1.7) tells us that $\widehat{K}f$ is the mass-shell restriction of the Fourier transform of $K\tilde{f}$ and thus $\widehat{K}f \in \mathcal{D}$.

Proposition II.2.6 : a) Let $K_1, \dots, K_n$ be regular kernels.

Let $\Delta_1, \dots, \Delta_{n-1}$ be a family of tempered distributions choosed in the set $\{\Delta_+, \Delta_-, \Delta_r, \Delta_a, \Delta_F, \Delta_{\bar{F}}\}$. Then the kernel $K_1 \Delta_1 \dots K_{n-1} \Delta_{n-1} K_n$ is regular.

b) $K_1 \Delta_1$ (resp $\Delta_1 K_1$) maps linearly and continuously $\mathcal{S}(\mathbb{R}^4)$ into $\mathcal{S}(\mathbb{R}^4)$ (resp $\mathcal{O}_M(\mathbb{R}^4)$ into $\mathcal{O}_M(\mathbb{R}^4)$).

Proposition II.2.7 : If one of the equations II.0.1 has a regular solution, this solution is unique.

Proof: For instance, in view of proposition II.2.6 and equation II.0.1, $(1 + i\Delta_F)$ is the inverse of $1 - A\Delta_F$.

Proposition II.2.8 : $\bar{I}$ (resp J_r) is a regular kernel (or W-I-P, or N-P kernel) if and only if $\bar{I}$ (resp J_a) is.

Proof : Indeed :

$$\bar{I}(A) = I(A^\dagger)^\dagger \quad J_r(A) = J_a(A^\dagger)^\dagger \quad (\text{II.2.4})$$

Definition II.2.9 : The external field will be called "physical" if it has only real components.

Proposition II.2.10 : a) The external field is physical if and only if

$$A = A^\dagger \quad (\text{II.2.5})$$

b) If the external field is real, and if equations II.0.1. have a regular solution, then these solutions obey :

$$J_r = J_a^\dagger \quad I^\dagger = \bar{I} \quad (\text{II.2.6})$$

Proof : See the proposition II.2.8 and the definition of A (eq.I.1.4)

Remark II.2.11 : Let G_r be defined by :

$$G_r = \Delta_r + \Delta_r J_r \Delta_r \quad (\text{II.2.7})$$

Then, one can easily convince oneself that G_r is the solution of the integral equation :

$$G_r = \Delta_r + \Delta_r A G_r = \Delta_r + G_r A \Delta_r \quad (\text{II.2.8})$$

Or equivalently G_r is the unique solution of the Klein-Gordon (cf. equation I.2.4) with external field, whose support is in $\{(x,y); (x,y) \in \bar{V}^+\}$ (Cauchy problem [6]). One can see that J_r can be reconstructed from G_r by the formula :

$$J_r = A + A G_r A \quad (\text{II.2.9})$$

II.3 UNITARITY

Proposition II.3.1 : Let $\bar{I}, \bar{I}, J_r, J_a$ be regular solutions of II.0.1
II.0.1 ; Then

a)

$$(\mathbb{1} + \hat{u} \bar{I}_s)^{-1} = (\mathbb{1} - \hat{u} \bar{I}_s)$$

a)

$$(\mathbb{1} + \hat{u} J_R)^{-1} = (\mathbb{1} - \hat{u} J_A)$$

b) (II.3.1)

on the domain $\mathcal{D}$.

b) If the external field is physical, then :

$(\mathbb{1} + \hat{u} \bar{I}_s)$ can be extended as an unitary operator on $\mathcal{H}_+ \oplus \mathcal{H}_- = \mathcal{H}$

$(\mathbb{1} + \hat{u} J_R)$ can be extended as a pseudo unitary operator on $\mathcal{H}$ with respect to the metric Γ .

Remark II.3.2 : The details of relation II.3.1 a) and b) are given in table I.

Remark II.3.3 : This proposition can be extended without change to the case of fields with integer spin. In the case of half-integers spin field, a) is also true. But in b) we have the following modification

$(\mathbb{1} + \hat{u} \bar{I}_s)$ is pseudo unitary with respect to Γ .

$(\mathbb{1} + \hat{u} J_R)$ is unitary.

Proof of the proposition II.3.1 :

Following Schwinger [3], and applying propositions II.2.4 and II.2.6 :

$$A = (1 - A \Delta) I = \bar{I} (1 - \Delta_{\bar{F}} A) \quad (\text{II.3.2})$$

Therefore, multiplying to the left by $1 + I \Delta_F$, and to the right by $1 + \Delta_F I$, we obtain :

$$0 = I - \bar{I} + I (\Delta_+ + \Delta_-) \bar{I} \quad (\text{II.3.3})$$

Adding the unit kernel to both sides we immediately obtain II.3.1 upon mass-shell restriction.

Now, if the external field is real one easily finds :

$$\bar{I}_S \subset I_S^* \quad J_A \subset J_R^* \quad I^{-1} \quad (\text{II.3.4})$$

where I is defined by II.1.12

Proposition II.3.4 : Let us assume that the external field is

physical :

a) if I is regular, then :

$$0 \leq \mathbb{1}_{++} - I_{+-} I_{-+}^* \leq \mathbb{1} \quad (\text{II.3.5})$$

and I_S is bounded.

b) If I is N-P then

$$0 \leq \det(\mathbb{1}_{++} - I_{+-} I_{-+}^*) \leq 1 \quad (\text{II.3.6})$$

c) If J is regular

$$\mathbb{1}_{++} - J_{r+-} J_{d+-} = \mathbb{1}_{++} - J_{r+-} J_{r+-}^* \geq \mathbb{1}_{++} \quad (\text{II.3.7})$$

Therefore this operator has a bounded inverse.

Remark II.3.5 : The result b) shows that proposition I.3.3 is likely to hold in a non-perturbative sense since :

$$0 \leq |\langle \Omega, S \Omega \rangle|^2 = \det(\mathbb{1}_{++} - \bar{I}_{+-} I_{-+}^*) \leq 1 \quad (\text{II.3.8})$$

TABLE I

1) $(\mathbb{1} + \vartheta I_S)(\mathbb{1} - \vartheta \bar{I}_S) = \mathbb{1}$	2) $(\mathbb{1} - \vartheta \bar{I}_S)(\mathbb{1} + \vartheta I_S) = \mathbb{1}$
$(\mathbb{1}_+ + \vartheta I_{++})(\mathbb{1}_+ - \vartheta \bar{I}_{++}) = \mathbb{1} - I_+ \bar{I}_{++}$	$(\mathbb{1} - \vartheta \bar{I}_{++})(\mathbb{1} + \vartheta I_{++}) = \mathbb{1} - \bar{I}_+ I_{++}$
$(\mathbb{1}_- + \vartheta I_{--})(\mathbb{1}_- - \vartheta \bar{I}_{--}) = \mathbb{1} - I_- \bar{I}_{--}$	$(\mathbb{1} - \vartheta \bar{I}_{--})(\mathbb{1} + \vartheta I_{--}) = \mathbb{1} - \bar{I}_- I_{--}$
$(\mathbb{1}_+ + \vartheta I_{++}) \bar{I}_{+-} = I_{+-} (\mathbb{1} - \vartheta \bar{I}_{++})$	$\bar{I}_{+-} (\mathbb{1} - \vartheta \bar{I}_{++}) = I_{+-} (\mathbb{1} + \vartheta I_{++})$
$(\mathbb{1}_- + \vartheta I_{--}) \bar{I}_{-+} = I_{-+} (\mathbb{1} - \vartheta \bar{I}_{--})$	$(\mathbb{1} - \vartheta \bar{I}_{--}) \bar{I}_{-+} = \bar{I}_{-+} (\mathbb{1} + \vartheta I_{--})$
3) $(\mathbb{1} + \vartheta J_R)(\mathbb{1} - \vartheta J_A) = \mathbb{1}$	4) $(\mathbb{1} - \vartheta J_A)(\mathbb{1} + \vartheta J_R) = \mathbb{1}$
$(\mathbb{1}_{++} + \vartheta J_{R++})(\mathbb{1}_{++} - \vartheta J_{A++}) = \mathbb{1} + J_{R+} J_{A-+}$	$(\mathbb{1} - \vartheta J_{A++})(\mathbb{1} + \vartheta J_{R++}) = \mathbb{1} + J_{A+-} J_{R-+}$
$(\mathbb{1} - \vartheta J_{R-})(\mathbb{1} + \vartheta J_{A--}) = \mathbb{1} + J_{R-+} J_{A+-}$	$(\mathbb{1} + \vartheta J_{A--})(\mathbb{1} - \vartheta J_{R--}) = \mathbb{1} + J_{A-+} J_{R+-}$
$(\mathbb{1} + \vartheta J_{R++}) J_{A+-} = J_{R+-} (\mathbb{1} + \vartheta J_{A--})$	$(\mathbb{1} + \vartheta J_{A--}) J_{R-+} = J_{A-+} (\mathbb{1} + \vartheta J_{R++})$
$(\mathbb{1} - \vartheta J_{R-}) J_{A-+} = J_{R-+} (\mathbb{1} - \vartheta J_{A++})$	$(\mathbb{1} - \vartheta J_{A++}) J_{R+-} = J_{A+-} (\mathbb{1} - \vartheta J_{R--})$

TABLE II

1) $(\mathbb{1} - \vartheta J_{R--})^{-1} = \mathbb{1} + \vartheta I_{--}$ $I_{+-} = J_{R-+} (\mathbb{1} + \vartheta I_{--})$	$J_{R++} - I_{++} = -\vartheta J_{R-+} \bar{I}_{+-} = -\vartheta I_{+-} J_{R-+}$ $J_{R-+} = (\mathbb{1} - \vartheta J_{R--}) I_{-+}$
2) $(\mathbb{1} + \vartheta J_{R++})^{-1} = \mathbb{1} - \vartheta I_{++}$ $\bar{I}_{+-} = (\mathbb{1} - \vartheta \bar{I}_{++}) J_{R-+}$	$J_{R--} - I_{--} = \vartheta J_{R-+} \bar{I}_{+-} = \vartheta \bar{I}_{-+} J_{R-+}$ $J_{R-+} = \bar{I}_{-+} (\mathbb{1} + \vartheta J_{R++})$
3) $(\mathbb{1} - \vartheta J_{A++})^{-1} = \mathbb{1} + \vartheta I_{++}$ $I_{+-} = (\mathbb{1} + \vartheta I_{++}) J_{A-+}$	$J_{A--} - I_{--} = -\vartheta J_{A-+} I_{+-} = -\vartheta I_{-+} J_{A-+}$ $J_{A-+} = I_{-+} (\mathbb{1} - \vartheta J_{A++})$
4) $(\mathbb{1} + \vartheta J_{A--})^{-1} = (\mathbb{1} - \vartheta \bar{I}_{--})$ $\bar{I}_{+-} = J_{A++} (\mathbb{1} - \vartheta \bar{I}_{--})$	$J_{A++} - \bar{I}_{++} = \vartheta J_{A-+} \bar{I}_{+-} = \vartheta \bar{I}_{-+} J_{A-+}$ $J_{A-+} = (\mathbb{1} + \vartheta J_{A--}) \bar{I}_{-+}$

Proof : We have only to remark that, since the external field is physical

$$I_{+-}^* = \bar{I}_{-+}$$

$$J_{a-+} = J_{r+-}^*$$

$$I_{++}^* = \bar{I}_{++}$$

$$J_{a++} = J_{r++}^*$$

By looking at Table I, proposition II.3.4 is proved.

II.4 RELATIONSHIP BETWEEN I AND J_r :

The main theorem is the following :

Theorem II.4.1 :

- 1) The equations (A) $J_r = A + A \Delta_a J_r = A + J_r \Delta_r A$ have a unique regular (resp W-N-P, N-P) solution such that $\Pi_+ J_{a-} J_{r-+}$ is invertible, if and only if the equations (B) $I = A + A \Delta_F I = A + I \Delta_F A$, $\bar{I} = A + A \Delta_{\bar{F}} \bar{I} = A + \bar{I} \Delta_{\bar{F}} A$ have a unique regular (resp W-N-P, N-P) solution such that $\Pi_+ I_{+-} \bar{I}_{-+}$ is invertible
- 2) If the external field is physical, equations (A) have a unique regular (resp W-N-P, N-P) solution if and only if equations (B) have a unique regular (resp W-P-N, N-P) solution such that $\Pi_- I_{+-} \bar{I}_{-+}^*$ is invertible

Proof : a) Let us suppose that $\bar{I}$ and J_r are regular solution of (A) and (B) respectively. Then we find

$$\begin{aligned} J_r &= (1 + J_r \Delta_r) A = (1 + J_r \Delta_r) (1 - A \Delta_F) \bar{I} \\ &= (1 + J_r \Delta_r) (1 - A \Delta_r + A \Delta_+) \bar{I} = (1 + J_r \Delta_+) \bar{I} \end{aligned}$$

$$J_r = I + J_r \Delta_+ \bar{I} \quad (\text{II.4.1})$$

By interchanging the different kernels $\mathcal{J}_a, \mathcal{J}_r, \mathcal{I}, \bar{\mathcal{I}}$ we obtain similar relations, whose details, when restricted on the mass-shell, are given in Table II.

b) Now let $\mathcal{J}_a$ be a regular solution of (A) ; If $\mathcal{A} + \mathcal{J}_{r+} \mathcal{J}_{a-+}$ is invertible, then, in view of the results of table I, $\mathcal{A} - \mathcal{E} \mathcal{J}_{r-}$ and $\mathcal{A} + \mathcal{E} \mathcal{J}_{r++}$ are invertible on the domain $\mathcal{D}$.

Therefore we can define an operator $\mathcal{I}_\mathcal{S}$ such that the relations of Table II hold.

We now define $\mathcal{I}(x, y)$ by :

$$\tilde{\mathcal{I}}(p, q) = \tilde{\mathcal{J}}_r(p, q) + 2\pi n \int \frac{d^3 r_1 d^3 r_2}{2\omega_{r_1} 2\omega_{r_2}} \tilde{\mathcal{J}}_r(p, -r_1) (\mathcal{A} - \mathcal{E} \mathcal{J}_{r-})(r_1, r_2) \tilde{\mathcal{J}}_r(-r_2, q) \quad (\text{II.4.2})$$

By construction and using the prop. II.2.5 $\mathcal{I}$ is regular and satisfies :

$$\hat{\mathcal{I}} = \mathcal{I}_\mathcal{S} ; \quad (\mathcal{A} + \mathcal{J}_r \Delta_+) \mathcal{I} = \mathcal{J}_r = \mathcal{I} (\mathcal{A} + \Delta_+ \mathcal{J}_r) \quad (\text{II.4.3})$$

Thus since

$$(\mathcal{A} - \mathcal{A} \Delta_r) (\mathcal{A} + \mathcal{J}_r \Delta_+) = (\mathcal{A} - \mathcal{A} \Delta_r) \quad (\text{II.4.4})$$

$\mathcal{I}$ is a regular solution of (B).

Then, we can remark that

$$(\mathcal{A} - \mathcal{I}_{+-} \bar{\mathcal{I}}_{-+})^{-1} = \mathcal{A} + \mathcal{J}_{r+} \mathcal{J}_{a-+} \quad (\text{II.4.5})$$

Now, if $\mathcal{J}_r$ is W-N-P (resp N-P), $\mathcal{I}$ is also W-N-P (resp N-P).

c) The converse can be shown by the same procedure.

d) The second part of the theorem is then trivial, by using proposition II.3.4.

Remark II.4.2 : The results 2) of theorem II.4.1 can be restated as follows :

If J_r is an N-P solution of (A) and if the external field is physical, then

$$0 \neq \det(1 - I_+ I_+^*) = \det(1 - I_{r+} I_{r+}^*)^{-1} \quad (\text{II.4.6})$$

Remark II.4.3 : Theorem II.4.1 can be extended without change to the case of arbitrary integer spin fields. In the case of half integer spin fields, $(I, \bar{I})$ have to be exchanged with (J_r, J_a) .

II.5 THE CLASSICAL S- MATRIX [5, 6, 8]

We suppose here that II.0.1 b) has a regular solution.

Proposition II.5.1 : Upon mass-shell restriction, the kernel

$S_{ee} = (1 - \Delta_a A)(1 - \Delta_r A)^{-1}$ defines a unique operator $\hat{S}_{ee}$ on $D \subset \mathcal{H}_+ \oplus \mathcal{H}_-$ (cf. prop. II.2.5) such that

$$\hat{S}_{ee} = 1 + i J_r \quad (\text{II.5.1})$$

Proof : Using II.0.1, we easily find :

$$S_{ee} = 1 + (\Delta_+ - \Delta_-) J_r \quad (\text{II.5.2})$$

This relation is sufficient to prove the proposition.

Proposition II.5.2 6 : If the external field is physical, then,

$1 + i J_{r++} = (\hat{S}_{ee})_{++}$ has a dense image.

Proof: Indeed, in this case, $1 + i J_{r,++}$ has a bounded inverse (cf proposition II.3.4)

Remark II.5.3 : The result of the previous proposition is a necessary and sufficient condition for the unicity of the out-vacuum as shown by Wightman (cf. [6] equation 4.12)
 This result can be extended to arbitrary integer spin fields. In the case of half integer spins, it might turn out to be wrong for strong external field [6, 15] .

Corollary II.5.4 : (Seiler [8])

- 1) $\hat{\mathcal{S}}_{\mathcal{C}}$ is pseudo unitary with respect to $\mathcal{I}$.
- 2) We have

$$(\mathcal{I}_{\mathcal{C}^+})^* = \overline{\mathcal{I}_{\mathcal{C}^-}} = (\hat{\mathcal{S}}_{\mathcal{C}})_{++}^{-1} (\hat{\mathcal{S}}_{\mathcal{C}})_{+-} \quad (\text{II.5.3})$$

Proof We have only to use proposition II.3.1, table I and the relation

$$\hat{\mathcal{S}}_{\mathcal{C}} = \mathbb{1} + \hat{\mathcal{G}}_{\mathcal{C}}$$

Remark II.5.5 : For integer spin fields, this corollary always holds. For half integer spin fields $\hat{\mathcal{S}}_{\mathcal{C}}$ is unitary [8] .
 (cf. remark II.3.3)

III - CONSTRUCTION OF THE BOGOLIUBOV S-OPERATOR IN FOCK SPACE.

III.1 DEFINITION OF $\mathcal{S}(I)$.

Following perturbation theory it seems natural to define the S-operator by :

$$\mathcal{S}_0(I) = \det(I - I_+ - I_-^*)^{1/2} \cdot \exp i \int \varphi^*(x) I(x, y) \varphi(y) dx dy \quad (\text{III.1 } 1)$$

If this expression is well defined, it is expected to be, up to a phase, a re-summation of the S-operator of the perturbation theory.

We shall, in fact show in this section that if I is an N-P kernel, $\mathcal{S}_0(I)$ is well defined on the dense domain of Fock space, generated by coherent states.

Then, if I is the N-P solution of the integral equation II.0.1, and we shall use an integral representation for $\mathcal{S}_0$ (section III.3) and show that $\mathcal{S}_0(I)$ can be continued as a unitary operator.

Next we shall show that $\mathcal{S}_0(I)$ satisfies the axioms of Bogoliubov [10,11] up to a phase (section III.4)

Finally we shall study the interpolating field, defined formally as in perturbation theory, and show that it is an operator valued tempered distribution, satisfying the Yang-Källén-Feldman equations [14,22]

It is unfortunate that this last part can only be generalized to fields with integer spin in view of the intensive use of coherent states in the computation of $\mathcal{S}_0(I)$

However, we should like to remark that the existence of $\mathcal{S}_0(I)$ as a unitary covariant and causal operator [10,11], up to a phase

can be shown, by using the representation theory of quasi free states, and the characterisation of unitarily implementable Bogoliubov transformation (cf for instance Berezin [24], Powers and Størmer [26], Manuceau and Verbeure [27], Van Daele and Verbeure [28]).

In particular, one could treat both cases : bosons fields as well as fermions fields.

But we have preferred to follow a more explicit method which can be used to prove the convergence, on coherent states, of the perturbation expansion for $\mathcal{S}_0(I)$ in the case of weak external fields.

III.2 COHERENT STATES

Definition III.2.1 (see App.1)

Let f be in $\mathcal{H}_+$, G be in $\mathcal{H}'_-$. The coherent state with wave functions f and G is defined as the vector of $\mathcal{F}$

$$\Phi_{f,G} = e^{\langle a^+, f \rangle + \langle G, b^+ \rangle} \Omega \quad (\text{III.2.1})$$

Let us note the formula

$$(\Phi_{f,G} | \Phi_{f',G'}) = e^{(f'|f) + (G', G)} \quad (\text{III.2.2})$$

Proposition [29] III.2.2

The vector space $\mathcal{D}_c$, spanned by the coherent states is a dense domain of $\mathcal{F}$. Any finite family of vectors of $\mathcal{D}_c$ is linearly independent.

Proposition III.2.3 :

The application $\Phi_f(f, G) \rightarrow \Phi_{f, G}$ from $\mathcal{H}_+ \times \mathcal{H}_-$ to $\mathcal{F}$ is an analytic function in $\mathcal{H}_+ \times \mathcal{H}_-$.

Proof : Let us recall [30] that a function $x \rightarrow f(x)$ from a complex Banach space E to a denumerable normed space F is analytic in an open set U of E if and only if (see [30], § 3.3)

- a) f is continuous
- b) f is locally bounded
- c) there exists a total set H on the dual space of F such that $\forall a \in U, \forall h \in H$, the function

$$t \rightarrow a + t h \rightarrow f(a + t h)$$

is holomorphic in a neighbourhood of zero in $\mathbb{C}$, $\forall u \in H$.

The result is then trivial if we use equation III.2.2

Proposition III.2.4 : Let P be a polynomial in two variables X and

Y . Then we have :

$$P(\varphi^*, \varphi) \circ \Phi_{f, G} = P(b^+_+ f, a^+_+ G) \Phi_{f, G} \quad (\text{III.2.3})$$

where $a^+_+ G$ (resp. $b^+_+ f$) is defined as the operator valued function defined on $\mathcal{H}_+$ (resp. on $\mathcal{H}_-$) $h \rightarrow a^+(h) + \langle G, h \rangle \mathbb{I}$ (resp. $H \rightarrow b^+(H) + \langle H, f \rangle \mathbb{I}$)

Proof: Using recursively the canonical commutation relations one finds :

$$a^-(F)^m b^-(g)^n \Phi_{f, G} = \langle F, f \rangle^m \langle G, g \rangle^n \Phi_{f, G} \quad (\text{III.2.4})$$

Equation III.2.3 follows by linear extension.

III.3 DEFINITION OF $S_o(I)$:

Propositions III.2.4 and III.2.2 allow us to define $e^{i\varphi^* I \varphi}$ as follows

$$e^{i\varphi^* I \varphi} \circ \Phi_{f, G} = e^{i(a^+_+ I_- b^+_+)} e^{i\langle G, I_- f \rangle} \Phi_{(a^+_+ I_-)f, G(a^+_+ I_-)} \quad (\text{III.3.1})$$

Therefore, it is sufficient to show the existence of $e^{i a^+_+ I_- b^+_+}$ as an operator densely defined on $\mathcal{D}_c$. The main result is the following :

Theorem III.3.1

- a) The vector $e^{i a^+_+ K b^+_+} \Phi_{f, G}$ belongs to $\mathcal{F}$ if and only if K is a H.S. operator with

$$\|K\|_{op} < 1 \quad (\text{II.3.2})$$

b) if $K, K' \in \hat{\mathcal{G}}_2 = \{K, K' \in \mathcal{L}(\mathcal{H}_+, \mathcal{H}_-), \|K\|_{op} < 1, \|K\|_{HS} < +\infty\}$
then

$$(e^{i a^+ K' b^+} \phi_{f', g'} | e^{i a^+ K b^+} \phi_{f, g}) = \frac{\exp \langle f'^*, g \rangle \begin{bmatrix} 1 & -i K \\ -i K'^* & 1 \end{bmatrix}^{-1} \begin{pmatrix} f \\ g'^* \end{pmatrix}}{\det(1 - K K'^*)} \quad (\text{III.3.3})$$

c) The mapping $F: (K, f, g) \rightarrow e^{i a^+ K b^+} \phi_{f, g}$ defined in $\hat{\mathcal{G}}_2 \times \mathcal{H}_+ \times \mathcal{H}'_-$ with values in F is analytic.

Let K be a H.S. operator. Then there exist [19,20] two orthonormal basis $(\varphi_i)_{i \in \mathbb{N}}$ in $\mathcal{H}_+$ and $(\Gamma_i)_{i \in \mathbb{N}}$ in $\mathcal{H}'_-$ such that :

$$\alpha) \quad -i K = \sum_{i \in \mathbb{N}} \lambda_i \varphi_i \otimes \Gamma_i \quad \text{converges in H.S. norm.}$$

$$\beta) \quad \lambda_i \geq 0 \quad \forall i \in \mathbb{N} \quad (\text{III.3.4})$$

$$\gamma) \quad \sum_{i \in \mathbb{N}} \lambda_i^2 = \text{tr}(K K^*) = \|K\|_{H.S.}^2 < +\infty$$

$$\delta) \quad \sup_{i \in \mathbb{N}} \lambda_i = \|K\|_{op}$$

Now, one defines a_i^+ and b_i^+ as follows :

$$a_i^+ = \langle a^+, \varphi_i \rangle \quad b_i^+ = \langle \Gamma_i, b^+ \rangle \quad (\text{III.3.5})$$

Then, one has also the following proposition :

Proposition III.3.2 : The following formula holds :

$$e^{i a^+ K b^+} = \lim_{N \rightarrow \infty} \prod_{i=1}^N \frac{1}{\lambda_i} \int_{\mathbb{C}} e^{-\frac{|z|^2}{\lambda_i}} e^{i(z a_i^+ + \bar{z} b_i^+)} \frac{d\bar{z} dz}{2i\pi} \quad (\text{III.3.6})$$

on the dense domain $\mathcal{D}_c$ of coherent states.

Proof of theorem III.3.1 :

i) Let us remark that if K is not a H.S. operator then $e^{\sum a^+ K b^+} \Phi_{f,G}$ does not belong to $\mathcal{F}$ as can be seen by computing the one particle - one antiparticle component.

ii) One will use the following notations :

$\mathcal{F}_i$ = Fock space spanned by (a_i^+, b_i^+) a)

Ω_i = vacuum of $\mathcal{F}_i$ b)

If $f \in \mathcal{H}_+$ and $G \in \mathcal{H}_-$ one puts :

$f_i = (\varphi_i | f)$ $G_i = (G | g_i)$ c)

$\Phi_i = \exp(-\lambda_i a_i^+ b_i^+ + f_i a_i^+ + G_i b_i^+) \Omega_i$ d) (III.3.7)

Hence :

$\Omega = \bigotimes_{i \in \mathbb{N}} \Omega_i$ $\mathcal{F} = \bigotimes_{i \in \mathbb{N}} \mathcal{F}_i$ e)

$e^{\sum a^+ K b^+} \Phi_{f,G} = \bigotimes_{i \in \mathbb{N}} \Phi_i$ f)

iii) One can easily see the following results :

- $\Phi_i \in \mathcal{F}_i$ if and only if $\lambda_i < 1$ and
- $\|\Phi_i\|^2 = (1 - \lambda_i^2)^{-1} \exp < (f_i, G_i), \begin{bmatrix} 1 & \lambda_i \\ \lambda_i & 1 \end{bmatrix}^{-1} \begin{pmatrix} f_i \\ G_i \end{pmatrix} >$ (III.3.8)

as can be seen from Lemma A.4.2 in Appendix 4.

- Hence, the vector :

$$e^{\sum a^+ K b^+} \Phi_{f,G} = \bigotimes_{i \in \mathbb{N}} \Phi_i$$

belongs to $\mathcal{F}$ only if $\|K\|_{op} < 1$, in view of the formulae III.3.4 d) and III.3.8

- The infinite product $\prod_{i \in \mathbb{N}} \|\Phi_i\|^2$ converges in $\mathbb{R}_+$ since the sequences $(\lambda_i)_{i \in \mathbb{N}}$, $(f_i)_{i \in \mathbb{N}}$, $(G_i)_{i \in \mathbb{N}}$ are in $\ell^2(\mathbb{N})$.

- Therefore, the vector $\bigotimes_{i \in \mathbb{N}} \Phi_i$ belongs to $\mathcal{F}$. Indeed, let $\Phi_{(N)}$ be defined as :

$$\Phi_{(N)} = \left(\bigotimes_{i=1}^N \Phi_i \right) \otimes \left(\bigotimes_{i=N+1}^{\infty} \Omega_i \right) \quad (III.3.9)$$

Then, if $M < N$, one has :

$$\|\Phi_{(N)} - \Phi_{(M)}\|^2 \leq \left(\prod_{i \in \mathbb{N}} \|\Phi_i\|^2 \right) \left(1 - \prod_{i=M+1}^N \|\Phi_i\|^2 \right) \quad (\text{III.3.10})$$

Thus $\Phi_{(N)}$ is a Cauchy sequence in $\mathcal{F}$, and is convergent.

iv) Let $\hat{\mathcal{G}}_2^F$ be the dense subset of $\hat{\mathcal{G}}_2$ whose elements are finite-rank operators. By Lemma A.4.2 (Appendix 4), it can be seen that III.3.3 holds if K and K' are in $\hat{\mathcal{G}}_2^F$. Putting $E = \mathcal{G}_2 \times \mathcal{H}_+ \times \mathcal{H}'_-$, $E^F = \hat{\mathcal{G}}_2^F \times \mathcal{H}_+ \times \mathcal{H}'_-$, and, $\alpha = (K, F, G) \in E$,

$$F(\alpha) = e^{a^+ K b^+} \Phi_{F, G}$$

$$H(\alpha', \alpha) = (F(\alpha') | F(\alpha)) \quad (\text{III.3.11})$$

One deduces that, $(\alpha', \alpha) \rightarrow H(\alpha', \alpha)$ is continuous as defined on $E^F \times E^F$, as can be seen from formula III.3.3. Therefore,

$H(\alpha', \alpha)$ can be uniquely continued as a continuous function on $E \times E$. On the other hand, one also has :

$$\|F(\alpha) - F(\alpha')\|^2 = H(\alpha, \alpha) + H(\alpha', \alpha') - H(\alpha', \alpha) - H(\alpha, \alpha') \quad (\text{III.3.12})$$

Thus, $\alpha \mapsto F(\alpha)$ is a continuous function on E . Now, F is clearly locally bounded, and we can see, from equation III.3.3 that :

$$\alpha \mapsto (\Phi_{F, G} | F(\alpha))$$

is analytic on E . Since $\mathcal{D}_c$ is total in $\mathcal{F}$, all the necessary and sufficient conditions hold for the analyticity of F (cf [30], § 3.3, or the proof of proposition III.2.3)

Proof of Proposition III.3.2 : By the previous proof (see eq. III.3.7) 10/11).

$$e^{\sum_{i=1}^{\infty} \lambda_i a_i^\dagger b_i^\dagger} = \lim_{N \rightarrow \infty} \prod_{i=1}^N e^{-\lambda_i a_i^\dagger b_i^\dagger} \quad (\text{III.3.14})$$

on a dense domain of coherent states.

It is then sufficient to prove (III.3.4) when

$$-a^\dagger K = \sum_i \lambda_i \varphi_i \otimes \Gamma_i$$

But it is easy to verify that, if $\lambda_i > 0$: (use lemma A.4.2 App.4) and theorem III.3.1)

$$\forall f' \in \mathcal{H}_+ \quad \forall G' \in \mathcal{H}'_-$$

$$(\phi_{f',G'} | e^{-\sum_i \lambda_i a_i^\dagger b_i^\dagger} \phi_{f,G}) = \frac{1}{\lambda_i} \int e^{-\frac{|\bar{z}|^2}{\lambda_i}} (\phi_{f',G'} | e^{\sum_i (a_i^\dagger z + b_i^\dagger \bar{z})} \phi_{f,G}) \frac{d\bar{z} dz}{2\pi} \quad (\text{III.3.15})$$

However, Schwarz's inequality implies :

$$|(\psi | e^{\sum_i (a_i^\dagger z + b_i^\dagger \bar{z})} \phi_{f,G})| \leq \|\psi\| e^{\frac{1}{2}(\|f + \sum_i z_i \varphi_i\|^2 + \|G + \sum_i \bar{z}_i \Gamma_i\|^2)} \quad (\text{III.3.16})$$

so that the integral converges in the strong sense if and only if

$$\lambda_i < 1.$$

III.4 BOGOLIUBOV 'S AXIOMS FOR $S_o(I)$ [10,11] .

Theorem III.4.1 : Let I be an N-P solution of the equation

$$I = A + I \Delta_F A = A + A \Delta_F I \quad (\text{III.4.1})$$

for a physical external field A , such that $\det(1 - I_+ I_+^*) \neq 0$

Then :

a) $S_o(I)$ densely defined on $\mathcal{D}_c$ by

$$S_o(I) \Phi_{(f,G)} =$$

$$\det(1 - I_+ I_+^*)^{\frac{1}{2}} e^{i\tilde{G} \langle I_+ f \rangle} e^{i\tilde{a}^+ I_+ b^+} \Phi_{(1 + i I_+)(f, G(1 + i I_-))} \quad (\text{III.4.2})$$

b) $S_o(I)$ can be continued as a unitary operator.

c) $S_o(I)$ is causal up-to-a-phase

d) $S_o(I)$ is relativistically covariant.

Proof: a) Since $\det(1 - I_+ I_+^*) > 0$, I_+ satisfies all the conditions of theorem III.3.1. Therefore a) is proved, since I is an N-P kernel (see section II).

b) By Eq. III.4.2 we find

$$(S_o(I) \Phi_{f'_0, G'_0} | S_o(I) \Phi_{f_0, G_0}) = (e^{i\tilde{a}^+ I_+ b^+} \Phi_{f'_0} | e^{i\tilde{a}^+ I_+ b^+} \Phi_{f_0}) \eta \det(1 - I_+ I_+^*) \quad (\text{III.4.3})$$

with

$$\begin{aligned} f' &= (1 + i I_+) f_0' & G' &= G'_0 (1 + i I_-) \\ f &= (1 + i I_+) f_0 & G &= G_0 (1 + i I_-) \end{aligned} \quad (\text{III.4.4})$$

and

$$\eta = \exp i(\langle G_0, I_+ f_0 \rangle - \overline{\langle G'_0, I_+ f'_0 \rangle}) \quad (\text{III.4.5})$$

By theorem III.3.1, we find

$$(\mathcal{S}_o(I)\phi_{f'_o G'_o} | \mathcal{S}_o(I)\phi_{f_o G_o}) = \exp \langle (f'_o)^*, G_o \rangle, B \begin{pmatrix} f_o \\ G_o^* \end{pmatrix} \rangle \quad (\text{III.4.6})$$

with

$$B = \begin{bmatrix} 0 & -i(I_+)^* \\ iI_+ & 0 \end{bmatrix} + \begin{bmatrix} \eta_+ iI_+^* & 0 \\ 0 & \eta_- iI_- \end{bmatrix} \begin{bmatrix} (\eta_+ I_+ I_+^*)^{-1} i(\eta_+ I_+ I_+^*)^{-1} I_+ \\ i(\eta_- I_- I_-^*)^{-1} I_-^* (\eta_- I_- I_-^*)^{-1} \end{bmatrix} \begin{bmatrix} \eta_+ iI_+ & 0 \\ 0 & \eta_- iI_-^* \end{bmatrix} \quad (\text{III.4.7})$$

With the results of table I, we find

$$B = \mathbb{1} \quad (\text{III.4.8})$$

Thus , $\forall f'_o, f_o \in \mathcal{H}_+$, $\forall G'_o, G_o \in \mathcal{H}'_-$:

$$(\mathcal{S}_o(I)\phi_{f'_o G'_o} | \mathcal{S}_o(I)\phi_{f_o G_o}) = (\phi_{f'_o G'_o} | \phi_{f_o G_o}) \quad (\text{III.4.9})$$

On the other hand, it can be seen without difficulty that

$$\mathcal{S}_o(\bar{I}) \subset \mathcal{S}_o(I)^* \quad (\text{III.4.10})$$

Thus $\mathcal{S}_o(I)$ is unitary

c) In order to show the causality property we have to compute $[\mathcal{S}_o(I_1)\mathcal{S}_o(I_2)]$. We shall use the integral representation given by prop. III.3.2 and the boundedness of $\mathcal{S}_o(I_1)$ for real external fields:

$$\mathcal{S}_o(I_1) s\text{-}\lim_{N \rightarrow \infty} A_N = s\text{-}\lim_{N \rightarrow \infty} \mathcal{S}_o(I_1) A_N \quad (\text{III.4.11})$$

if A_N is a sequence of operators.

On the other hand, the strong convergence of integral III.4.5 allows us to push $\mathcal{S}_o(I_i)$ under the integral sign. (III.4.12)

By computation of several Gaussian integrals we find, $\forall f_o \in \mathcal{H}_r, \forall G_o \in \mathcal{H}'_r$:

$$\mathcal{S}_o(I_1) \mathcal{S}_o(I_2) \Phi_{f_o G_o} = \eta \mathcal{S}_o(I_3) \Phi_{f_o G_o} \quad (\text{III.4.13})$$

with

$$\eta = \frac{\det(\mathbb{1}_+ - \mathbb{I}_{1+} - \mathbb{I}_{2+}^*)^{1/2} \det(\mathbb{1}_+ - \mathbb{I}_{2+} - \mathbb{I}_{1+}^*)^{1/2}}{\det(\mathbb{1}_+ - \mathbb{I}_{3+} - \mathbb{I}_{3+}^*)^{1/2} \det(\mathbb{1}_+ + \mathbb{I}_{1+} - \mathbb{I}_{2+})} \quad (\text{III.4.14})$$

and

$$\mathbb{1}_+ \circ \mathbb{I}_{3++} = (\mathbb{1}_+ \circ \mathbb{I}_{1++}) (\mathbb{1}_+ - \mathbb{I}_{1+-} \mathbb{I}_{2+-})^{-1} (\mathbb{1}_+ + \mathbb{I}_{2++}) \quad \text{a)}$$

$$\mathbb{1}_+ \circ \mathbb{I}_{3--} = (\mathbb{1}_+ \circ \mathbb{I}_{2--}) (\mathbb{1}_+ - \mathbb{I}_{2+-} \mathbb{I}_{1+-})^{-1} (\mathbb{1}_+ + \mathbb{I}_{1--}) \quad \text{b)}$$

(III.4.15)

$$\mathbb{I}_{3-+} = \mathbb{I}_{2-+} + (\mathbb{1}_+ \circ \mathbb{I}_{2--}) (\mathbb{1}_+ - \mathbb{I}_{1+-} \mathbb{I}_{2+-})^{-1} \mathbb{I}_{1-+} (\mathbb{1}_+ \circ \mathbb{I}_{2++}) \quad \text{c)}$$

$$\mathbb{I}_{3+-} = \mathbb{I}_{1+-} + (\mathbb{1}_+ \circ \mathbb{I}_{1++}) (\mathbb{1}_+ - \mathbb{I}_{2+-} \mathbb{I}_{1+-})^{-1} \mathbb{I}_{2+-} (\mathbb{1}_+ \circ \mathbb{I}_{1--}) \quad \text{d)}$$

With the help of the results of table I, it can be seen that $\mathbb{1}_+ \circ \mathbb{I}_{3\mathcal{S}}$ is also unitary and that η is a complex number with modulus 1.

Now, if we construct $\mathcal{J}_{3,\mathcal{R}}$ from $\mathbb{I}_{3,\mathcal{S}}$ by the formulae given in table II, we find as a consequence of theorem II.4.1.

$$(\mathbb{1}_+ \circ \mathcal{J}_{3,\mathcal{R}}) = (\mathbb{1}_+ \circ \mathcal{J}_{1,\mathcal{R}}) (\mathbb{1}_+ \circ \mathcal{J}_{2,\mathcal{R}}) \quad (\text{III.4.16})$$

This formula suggests that $\mathbb{I}_3$ is associated to a problem in which the corresponding retarded kernel $\mathcal{J}_{r3}$ is

$$\mathcal{J}_{r3} = \mathcal{J}_{r1} + \mathcal{J}_{r2} + \mathcal{J}_{r1} \Delta \mathcal{J}_{r2} \quad (\text{III.4.17})$$

where $\mathcal{J}_{ri} (i=1,2)$ is the solution of

$$\bar{\mathcal{J}}_{ri} = A_i + A_i \Delta_r \mathcal{J}_{ri} = A_i + \mathcal{J}_{ri} \Delta_r A_i \quad (\text{III.4.18})$$

Let us now assume that $([10]_{p_1}, [11])$

$$\text{supp}(v_1, A_{\mu_1}) \cap (\bar{V}^- + \text{supp}(v_2, A_{\mu_2})) = \emptyset \quad (\text{III.4.19})$$

Then we easily find

$$A_1 \Delta_a A_2 = J_{r,1} \Delta_a J_{r,2} = J_{r,1} \Delta_a A_2 = A_1 \Delta_a J_{r,2} = 0 \quad (\text{III.4.20})$$

Since we have the same formula by simultaneously interchanging 1 and 2, Δ_a and Δ_r ,

$$J_{1,r} \Delta J_{2,r} = J_{1,r} \Delta_r \Delta_{2,r} \quad (\text{III.4.21})$$

and

$$J_{3,r} = (A_1 + A_2)(1 + \Delta_r J_{3,r}) = (1 + J_{3,r} \Delta_r)(A_1 + A_2) \quad (\text{III.4.22})$$

Therefore, I_3 is a kernel associated with the external field

$$((v_1 + v_2), (A_{\mu_1} + A_{\mu_2})) \quad (\text{III.4.23})$$

which proves causality up-to-a phase.

d) The covariance of $\mathcal{S}_0(I)$ is easy to show. Recall that

$$(v, A_\mu)(x) = (v, \Lambda_\mu^\nu A_\nu)(\Lambda^{-1}(x-a)) \quad (\text{III.4.24})$$

Then :

$$(a, \Lambda) I(x, y) = I(\Lambda^{-1}(x-a), \Lambda^{-1}(y-a)) \quad (\text{III.4.25})$$

If $U_\epsilon(a, \Lambda)$ is the representation of the Poincaré group in $\mathcal{H}_\epsilon$, it is clear from II.1.7, that in momentum space :

$$(a, \Lambda) I_{\epsilon\epsilon'} = U_\epsilon(a, \Lambda) I_{\epsilon\epsilon'} U_{\epsilon'}(a, \Lambda)^{-1} \quad (\text{III.4.26})$$

Therefore, $\det(\mathbb{1} - I_{+-} I_{+-}^*)$ is invariant, and

$$:e^{\int \varphi^*(a, \Lambda) I \varphi} : = U(a, \Lambda) :e^{\int \varphi^* I \varphi} : U(a, \Lambda)^{-1} \quad (\text{III.4.27})$$

Theorem III.4.2 :

i) The mapping $I \rightarrow S_o(I)$ is strongly continuous when we norm the space of N-P kernel with

$$\|I\|_{N.P} = \max \{ \|I_{++}\|_{op}, \|I_{--}\|_{op}, \|I_{+-}\|_{H.S}, \|I_{-+}\|_{H.S} \}$$

ii) The mapping $(I, f, G) \rightarrow \det(\mathbb{1} - I_{+-} I_{+-}^*)^{-1/2} S_o(I) \Phi_{f, G}$ is analytic in the domain

$$f \in \mathcal{H}_+, G \in \mathcal{H}_-, I_{+-} \in \hat{\mathcal{G}}_2, I_{-+} \in \hat{\mathcal{G}}_2, I_{++} \in \mathcal{L}(\mathcal{H}_+), I_{--} \in \mathcal{L}(\mathcal{H}_-)$$

Proof : a) it is well known that [20]

$$I_{+-} \rightarrow \det(\mathbb{1} - I_{+-} I_{+-}^*)^{1/2}$$

is continuous with respect to the H.S norm.

b) From the analyticity [30] of

$$\begin{array}{ccc} (I_{++}, f) & \longrightarrow & I_{++} f \\ \mathcal{L}(\mathcal{H}_+) \times \mathcal{H}_+ & & \mathcal{H}_+ \end{array}$$

and from theorem III.3.1, ii) follows

c) i) is then a simple consequence of ii).

III.5 THE INTERPOLATING FIELDS .

Theorem III.5.1

Let ψ be the field formally defined by:

$$\psi(x) = :e^{i\int \varphi^* \bar{I} \varphi} : \int d^4 y (1 + \Delta_F I)(x, y) \varphi(y) e^{i\int \varphi^* I \varphi} : \quad (\text{III.5.1})$$

- a) $\psi(f) = \int \psi(x) f(x) d^4 x$ is densely defined on $\mathcal{D}_c$
 $\forall f \in \mathcal{S}(\mathbb{R}^4)$
- b) $\psi(x)$ is an operator valued distribution, [32], which is a solution of the Yang-Feldman-Källén equation, on the dense domain $\mathcal{D}_c$.

Proof: a) First of all, $f \rightarrow \int d^4 x f(x) (1 + \Delta_F I)(x, y)$ is a continuous linear map from $\mathcal{S}(\mathbb{R}^4)$ to $\mathcal{S}'(\mathbb{R}^4)$, and therefore $(1 + \Delta_F I)\varphi$ is a good operator valued distribution.

On the other hand $:e^{i\int \varphi^* \bar{I} \varphi} :$ can be continued as a bounded operator, as follows from theorem III.4.1. Therefore, we have to show that if $f \in \mathcal{H}_+$, $g \in \mathcal{H}'_-$, then

$$:\varphi(h) e^{i\int \varphi^* \bar{I} \varphi} : \Phi_{f, g} \in \mathcal{F} \quad \forall h \in \mathcal{S}(\mathbb{R}^4) \quad (\text{III.5.2})$$

By using properties of the coherent states (proposition III.2.4) it is sufficient to show that

$$e^{i a^\dagger I_- b^\dagger} \Phi_{f, g} \quad (\text{III.5.3})$$

is in the domain of $b^\dagger$ (or $a^\dagger$) for any wave function f (or g). In order to see this, recall that the domain of $b^\dagger$ (or $a^\dagger$) is the set of vectors Ψ such that

$$\sum_{s, r} s \|\Psi\|_{rs}^2 < +\infty \quad (\text{or } \sum_{s, r} r \|\Psi\|_{rs}^2 < +\infty) \quad (\text{III.5.4})$$

Now, let us put for $B > 0$

$$\|\Psi\|_B^2 = \sum_{r,s} B^{r+s} \|\Psi\|_{rs}^2 \quad (\text{III.5.5})$$

We see that

$$\|e^{\hat{a}^\dagger I_+ b^\dagger} \Phi_{f,G}\|_B^2 = \|e^{\hat{a}^\dagger B a^\dagger I_+ b^\dagger} \Phi_{\sqrt{B}f, \sqrt{B}G}\|_B^2 \quad (\text{III.5.6})$$

which, by theorem III.3.1 is finite if and only if

$$\|I_+\|_{op} < 1/B \quad (\text{III.5.7})$$

In view of the condition $\|I_+\|_{op} < 1$, there exists $B > 1$ such that $\|I\|_{op} < \frac{1}{B}$. Therefore

$$B \rightarrow \|e^{\hat{a}^\dagger I_+ b^\dagger} \Phi_{f,G}\|_B^2 \quad (\text{III.5.8})$$

is analytic in an open domain containing the closed set $B \leq 1$. Therefore

$$\left. \frac{d}{dB} \|e^{\hat{a}^\dagger I_+ b^\dagger} \Phi_{f,G}\|_B \right|_{B=1} = \sum_{r,s} (r+s) \|\dots\|_{rs}^2 < +\infty \quad (\text{III.5.9})$$

Remark III.5.2

By the same method we also prove that $e^{\hat{a}^\dagger I_+ b^\dagger} \Phi_{f,G}$ is in the domain of $(a^\dagger)^m (b^\dagger)^n \quad \forall m, \forall n$

b) Now it is easy to compute the matrix elements of $\psi(x)$ between coherent states. We have only to use theorem III.3.1 and table I. We find that

$$\begin{aligned} (1 - \Delta_r A) (\Phi_{f',G'} | \psi(x) \Phi_{f,G}) &= (\Phi_{f',G'} | \varphi(x) \Phi_{f,G}) \\ (1 - \Delta_a A) (\Phi_{f',G'} | \psi(x) \Phi_{f,G}) &= (\Phi_{f',G'} | \varphi_{out}(x) \Phi_{f,G}) \end{aligned} \quad (\text{III.5.10})$$

which are the Yang-Feldman-Källén equations. [14,22]

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- APPENDIX I

FOCK SPACE AND TWO POINT FUNCTIONS

1°) Let $\mathcal{H}$ be a Hilbert space. We denote by $\mathcal{H}'$ its dual. By Riez's theorem [18] we can find an antiisomorphism

$$\mathcal{J}: \alpha \longrightarrow \alpha^*$$

from $\mathcal{H}$ to $\mathcal{H}'$, defined by

$$\forall y \in \mathcal{H} \quad \langle \alpha^*, y \rangle = (\alpha | y) \quad \text{A.1.1}$$

$\mathcal{H}'$ is a Hilbert space with inner product :

$$(\alpha' | y')_{\mathcal{H}'} = (\mathcal{J}' y' | \mathcal{J}' \alpha)_{\mathcal{H}} \quad \text{A.1.2}$$

We shall denote also $\mathcal{J}' \alpha'$ by α'^* , which is not confusing, by eq. A.1.2. .

2°) In order to construct the Fock space for charged bosons, we need two Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$. We shall define $\mathcal{F}$ by

$$\mathcal{F} = \bigoplus_{r,s} (\mathcal{H}_+)^{V_r} \otimes (\mathcal{H}_-)^{V_s} \quad \text{A.1.3}$$

Here V means "symmetrical tensor product".

If $\Phi \in \mathcal{F}$ we can write

$$\Phi = (\Phi_{ns})_{n,s \in \mathbb{N}^2} \quad \Phi_{n,s} \in \mathcal{H}_+^{V_r} \otimes \mathcal{H}_-^{V_s} \quad \text{A.1.4}$$

We put :

$$\Omega = (\Omega_{r,s})_{(r,s) \in \mathbb{N}^2} \quad \Omega_{0,0} = 1 \quad \Omega_{r,s} = 0 \text{ if } (r,s) \neq 0 \quad \text{A.1.5}$$

3°) Creation and annihilation operators for particles or antiparticles are defined as follows

$$\forall F \in \mathcal{H}'_+, \forall f \in \mathcal{H}_+, \forall G \in \mathcal{H}'_-, \forall g \in \mathcal{H}_-$$

$$a^-(F)\Omega = 0 \quad b^-(g)\Omega = 0 \quad \text{a)}$$

$$[a^\pm, b^\pm] = 0 \quad \text{A.1.6}$$

$$[a^-(F), a^-(f)] \subset \langle F, f \rangle \mathbb{1}_{\mathcal{F}}, \quad [b^-(g), b^-(G)] \subset \langle G, g \rangle \mathbb{1}_{\mathcal{F}} \quad \text{b)}$$

$$a^+(f) = a^-(f^*)^* \quad b^+(G) = b^-(G^*)^* \quad \text{c)}$$

4°) The field is defined if we can find four linear continuous maps defined on the space of test function $\mathcal{T}(\mathbb{R}^4)$

$$u_\pm, u'_\pm \quad \text{such that } [6, 33, 34]$$

$$\text{Ran}(u_\pm) \subset \mathcal{H}_\pm \quad \text{Ran}(u'_\pm) \subset \mathcal{H}'_\pm \quad \text{A.1.7}$$

Then $\forall h \in \mathcal{T}(\mathbb{R}^4)$

$$\varphi(h) = a^-(u'_+(h)) + b^+(u'_-(h)) \quad \text{a)}$$

$$\varphi^*(h) = a^+(u_+(h)) + a^-(u_-(h)) \quad \text{b)} \quad \text{A.1.8}$$

with the condition (if $h \rightarrow h^*$ is the natural involution in $\mathcal{T}(\mathbb{R}^4)$).

$$u_\pm(h)^* = u'_\pm(h^*) \quad \text{A.1.9}$$

For more complete details about covariance see for instance [6, 33, 34].

5°) For a scalar field we choose $\mathcal{H}_+ = \mathcal{H}_- = \mathcal{H}'_+ = \mathcal{H}'_-$
 $= \mathcal{L}^2(H_m, d\Omega_m(p))$ where

$$H_m = \{p \in \mathbb{R}^4; p^0 = \sqrt{\vec{p}^2 + m^2} = \omega_p\} \quad d\Omega_m(p) = \frac{d^3\vec{p}}{2\omega_p} \quad \text{A.1.10}$$

We have to put :

$$\begin{aligned} \forall h \in \mathcal{S}(\mathbb{R}^4) \\ u_+(h) = u'_-(h) = p \rightarrow (2\pi)^{-3/2} \int e^{-i\vec{p} \cdot \vec{x}} h(x) d^4x \Big|_{p^0 = \omega_p} \\ u_-(h) = u'_+(h) = p \rightarrow (2\pi)^{-3/2} \int e^{i\vec{p} \cdot \vec{x}} h(x) d^4x \Big|_{p^0 = \omega_p} \end{aligned} \quad \text{A.1.11}$$

Then

$$[\varphi(h), \varphi^*(k)] \subset i \left(\int h(x) k(y) \Delta(x-y) dx dy \right) \mathbb{1}_{\mathcal{G}} \quad \text{A.1.12}$$

with $\Delta = \Delta_+ - \Delta_-$ and

$$\Delta_{\pm}(\xi) = \frac{-i}{(2\pi)^{3/2}} \int_{H_m} e^{\pm i\vec{p} \cdot \vec{\xi}} d\Omega_m(p) \quad \text{A.1.13}$$

We then define Δ_r and Δ_a by

$$\Delta = \Delta_r - \Delta_a \quad \text{a)}$$

$$\text{Supp } \Delta_r \subset \bar{V}^+, \text{ Supp } \Delta_a \subset \bar{V}^- \quad \text{b)} \quad \text{A.1.14}$$

$$(\square + m^2) \Delta_a = \delta \quad \text{c)}$$

- APPENDIX 2

WICK'S THEOREMS

 1) Weak Wick's theorem [17]:

Let φ be a free scalar field. Then

$$\prod_{i=1}^n \frac{\varphi^{a_i}(x_i)}{a_i!} = \sum_{0 \leq \beta_i \leq a_i} (\Omega) \prod_{i=1}^n \frac{\varphi^{\beta_i}(x_i) \Omega}{\beta_i!} : \prod_{i=1}^n \frac{\varphi^{a_i - \beta_i}(x_i)}{(a_i - \beta_i)!} : \quad \text{A.2.1}$$

Let now $\lambda = (\lambda_i(x))_{i \in \mathbb{N}}$ be a family of test functions, a finite number only are non zero. We define

$$\mathcal{L}^{(a)}(x; \lambda) = \sum_{\beta \geq 0} \lambda_{\alpha+\beta}(x) \frac{\varphi^{\beta}(x)}{\beta!} \quad \text{A.2.2}$$

With $\mathcal{L}^{(0)} = \mathcal{L}$ A.2.1 can be written in the formal powers serie sense with respect to λ :

$$e^{i \int d^4x \mathcal{L}(x; \lambda)} = : e^{\sum_{\alpha \geq 0} \int d^4x \mathcal{L}^{(\alpha)}(x; \lambda) \frac{\delta}{\delta \lambda_{\alpha}(x)}} : (\Omega) e^{i \int d^4x \mathcal{L}(x; \lambda') \Omega} \quad \text{A.2.3}$$

Formulae A.2.1, A.2.3 can be generalized without difficulty, by replacing α by a multiindex $(\alpha_1, \dots, \alpha_N)$ if $\varphi = (\varphi_1, \dots, \varphi_N)$.

If some components describe a fermion field, the corresponding "coupling" constants λ_{α_i} have to take values in a formal Grassman algebra to give the correct signs in A.2.1.

2. STRONG WICK'S THEOREM

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an n -index. Let $\Gamma_{(\alpha)}$ be the set of graphs with vertices $1, 2, \dots, n$ such that α_i lines meet at vertex " i ".

If ℓ is a line of the graph G , we denote by $u(\ell)$ and $v(\ell)$ its extremity, with this convention :

$$u(\ell) < v(\ell)$$

Then we have

$$(\Omega | \prod_{i=1}^n \frac{\phi^{d_i}(x_i)}{d_i!} \Omega) = \sum_{G \in \Gamma_{(\alpha)}} \prod_{\ell \in G} \Delta_+(x_{u(\ell)} - x_{v(\ell)}) \quad \text{A.2.4}$$

The right-hand side is defined in $\mathcal{S}'(\mathbb{R}^4)$ as the boundary value of a function analytic in the tube :

$$\text{Im}(x_i - x_{i+1}) \in V^+ \quad i = 1, \dots, n-1 \quad \text{A.2.5}$$

If G is a tree-graph, the product $\prod_{\ell \in G} \dots$ is a simple tensor product of distribution [11,25]

Formula A.2.4 can be easily generalised for field with several components by introducing several kinds of lines and other "propagators".

3. WICK'S THEOREM FOR CHRONOLOGICAL PRODUCT [11]

One formally can replaced product of fields by chronological product in A.2.1 or A.2.4. Epstein and Glaser [11] have shown that it is possible to define a T-product such that A.2.3 holds, and such that A.2.4 holds for tree graphs (see also Bogoliubov and Chirkov [10]).

APPENDIX 3

PROOF OF THEOREM I.1.1

1°) We want to compute $S(g, \underline{J})$ for the following lagrangean density :

$$\mathcal{L}_I(x, g, \underline{J}) = v : \varphi^* \varphi : + i A_\mu : \varphi^* \overleftrightarrow{\partial} \varphi : + J \varphi + \bar{J} \varphi^* + J_\mu \partial^\mu \varphi + \bar{J}_\mu \partial^\mu \varphi^* \quad A.3.1$$

with

$$g = (v, A_\mu) \quad \underline{J} = (J, \bar{J}, J_\mu, \bar{J}_\mu) \quad A.3.2$$

It is more convenient to write the field in the Petiau-Duffin-Kemmer [35] form :

$$\hat{\psi} = \begin{bmatrix} \varphi \\ \partial^\mu \varphi \end{bmatrix} \quad \hat{\psi}^* = [\varphi^*, \partial^\mu \varphi^*] \quad \hat{j} = [J, J_\mu] \quad \bar{\hat{j}} = \begin{bmatrix} \bar{J} \\ \bar{J}_\mu \end{bmatrix} \quad A.3.3$$

and

$$a = \begin{bmatrix} g & i A_\mu \\ i A_\mu & 0 \end{bmatrix}$$

Then, A.3.1 becomes :

$$\mathcal{L}_I(x, g, \underline{J}) = \hat{j} \cdot \hat{\psi} + \hat{\psi}^* \bar{\hat{j}} + : \hat{\psi}^* a \hat{\psi} : \quad A.3.4$$

2°) We can apply the weak Wick theorem (App. 2 (A.2.3)) :

$$S(g, \underline{J}) = T \left(\exp i \int \mathcal{L}_I(x, g, \underline{J}) d^4x \right) = \\ : e^{i \int \mathcal{L}_I(x, g, \underline{J}) d^4x + \int (\hat{j} + \hat{\psi}^* a) \frac{\delta}{\delta \hat{j}(x)} + \frac{\delta}{\delta \bar{\hat{j}}(x)} (a \hat{\psi} + \bar{\hat{j}})(x) d^4x} : \left(\Omega, S(g, \underline{J}) \Omega \right) \Big|_{\underline{J}=0} \quad A.3.5$$

(Use also Taylor's formula).

On the other hand, by using the strong Wick theorem for tree graphs (App. 2 §2 and 3) we can find :

$$\frac{\delta}{\delta j(x)} (\Omega | S(q, \underline{J}) \Omega) = i \int G_F(x, y) \bar{J}(y) dy (\Omega | S(q, \underline{J}) \Omega) \quad \text{a)} \quad \text{A.3.6}$$

$$\frac{\delta}{\delta \bar{J}(y)} (\Omega | S(q, \underline{J}) \Omega) = i \int \bar{j}(x) G_F(x, y) dx (\Omega | S(q, \underline{J}) \Omega) \quad \text{b)}$$

where G_F is defined by :

$$G_F = S_F + S_F a G_F = S_F + G_F a S_F \quad \text{a)} \quad \text{A.3.7}$$

$$S_F(x-y) = (\Omega | T(\hat{\psi}(x) \hat{\psi}^*(y)) \Omega) \quad \text{b)}$$

Therefore (see Schwinger^[3]) the unique solution of A.3.6 is

$$(\Omega | S(q, \underline{J}) \Omega) = (\Omega | S(q, 0) \Omega) e^{i \int j G_F \bar{J}} \quad \text{A.3.8}$$

Omitting integrations, we can write, by Taylor's formula

$$S(q, \underline{J}) = e^{i \int j G_F \bar{J}} : e^{i \int \hat{\psi}^* \bar{J} \hat{\psi} + j (1 - S_F a) \hat{\psi} + \hat{\psi}^* (1 - a S_F) \bar{J}} : \dots \quad \text{A.3.9}$$

$$\dots (\Omega | S(q, 0) \Omega)$$

$$\bar{J} = a + a G_F a$$

3°) Now, we have to introduce the distribution kernel $I(x, y)$ defined by

$$\int \varphi^* I \varphi = \int \hat{\psi}^* \bar{J} \hat{\psi} \quad \text{A.3.10}$$

In order to compute $\bar{I}$ we have to compute $\mathcal{S}'_F$. Indeed power counting tells us that [11]

$$(\Omega | T(\partial_\mu \varphi(x) \partial_\nu \varphi^*(y)) \Omega)$$

is defined up to a counter-term of the form

$$a g_{\mu\nu} \delta(x-y) \quad a \in \mathbb{R} \quad \text{A.3.11}$$

By choosing

$$(\Omega | T(\partial_\mu \varphi(x) \partial_\nu \varphi^*(y)) \Omega) = \frac{1}{\epsilon} (\partial_\mu \partial_\nu \Delta_F + g_{\mu\nu} \delta)(x-y) \quad \text{A.3.12}$$

we find

$$I = A + A \Delta_F I = A + I \Delta_F A \quad \text{A.3.13}$$

with

$$A(x,y) = [v(x) + A_\mu A^\mu(x)] \delta(x-y) + i [A_\mu(x) + A_\mu(y)] \partial^\mu \delta(x-y) \quad \text{A.3.14}$$

And therefore

$$\mathcal{S}(g,0) = \mathcal{S}(g) = : e^{i \int \varphi^* I \varphi} : (\Omega | \mathcal{S}(g) \Omega) \quad \text{A.3.15}$$

Remark

This choice of renormalisation is associated with the minimal coupling for the two points Green's function.

4°) In order to compute the interpolating field we have only to derive A.3.9

$$\mathcal{S}(g, I)^{-1} \frac{\delta \mathcal{S}}{\delta J(x)}(g, I) \Big|_{I=0} = \psi(x) \quad \text{A.3.16}$$

We must remark that :

$$\frac{\delta}{\delta J(x)} \int (1 - \Delta_F A)^{-1} \hat{\psi} = (1 - \Delta_F A)^{-1} \varphi = (1 + \Delta_F I) \varphi \quad \text{A.3.17}$$

because

$$(1 + \Delta_F I)(1 - \Delta_F A) = (1 - \Delta_F A)(1 + \Delta_F I) = 1 \quad \text{A.3.18}$$

Therefore proposition I.4.1 is proved.

Remark :

The same calculation can be performed for any free field and any quadratic lagrangean. It gives us a "preferred" choice of renormalisation for fields with high spins.

APPENDIX 4

GAUSSIAN INTEGRALS

Let $E = \mathbb{C}^n$ with its canonical Hilbert structure. If $\alpha \in E$, let $\alpha^* \in E'$ the linear form on E defined by (cf Appendix 1. §2)

$$\langle \alpha^*, y \rangle = (\alpha | y) \quad \forall y \in E \quad \text{A.4.1}$$

Then:

Lemma A.4.1

The following formula holds, provided

$$A + A^* > 0. \quad \text{a)}$$

$$e^{-\langle \alpha, A \beta \rangle} \det(A) = \int_E e^{-\langle z^*, A z \rangle + i \langle \alpha, z \rangle + i \langle z^*, \beta \rangle} d\rho_n(z, z^*) \quad \text{b)} \quad \text{A.4.2}$$

Here $\alpha \in E'$ and $\beta \in E$.

$$d\rho_n(z, z^*) = d^n(\operatorname{Re} z) d^n(\operatorname{Im} z) \pi^{-n} \quad \text{c)}$$

Remark : If $A + A^* > 0$ then A is regular.

Now let $(\varphi_i)_{i \in \mathbb{N}^+}$ and $(\Gamma_i)_{i \in \mathbb{N}^+}$ be two orthogonal basis in $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively. Let a_i^+ , b_i^+ be defined by

$$a_i^+ = a^+(\varphi_i) = \langle a^+, \varphi_i \rangle, \quad b_i^+ = b^+(\Gamma_i) = \langle \Gamma_i, b^+ \rangle \quad \text{A.4.3}$$

Then we have the following result.

Lemma A.4.2

Let λ, λ' be in $\mathcal{L}(\mathcal{C}^n)$, α, α' be in $\mathcal{C}^n$, β, β' in $\mathcal{C}^n$. We put :

$$\Psi(\lambda, \alpha, \beta) = \exp \left[\sum \lambda_{ij} a_i^\dagger b_j^\dagger + \sum (\alpha_i a_i^\dagger + \beta_i b_i^\dagger) \right] \Omega \quad \text{A.4.4}$$

Then we have

a) $\Psi(\lambda, \alpha, \beta) \in \mathcal{F}$ if and only if $0 \leq \lambda \lambda^* < \mathbb{1}$

b)

$$(\Psi(\lambda', \alpha', \beta') | \Psi(\lambda, \alpha, \beta)) = \frac{\exp \left\langle \alpha'^* / \beta' \left[\begin{array}{cc} \mathbb{1}_n & -\lambda \\ -\lambda'^* & \mathbb{1}_n \end{array} \right] \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\rangle}{\det_n (\mathbb{1} - \lambda \lambda'^*)} \quad \text{A.4.5}$$

Proof : To establish formula (IV.4.5) we can use Bargman's representation [36] of Fock space with a finite number of degree of freedom.

Then, $\Psi(\lambda, \alpha, \beta)$ is represented by the analytic entire function

$$\hat{\Psi}_{\lambda, \alpha, \beta}(\xi, \eta) = \exp \left(\langle \xi, \lambda \eta \rangle + \langle \xi, \alpha \rangle + \langle \beta, \eta \rangle \right) \quad \text{A.4.6}$$

with $\xi \in E'$, $\eta \in E$.

The scalar product is given by

$$(\hat{\Psi}', \hat{\Psi}) = \int e^{-\langle \xi, \xi^* \rangle - \langle \eta^*, \eta \rangle} \hat{\Psi}'(\xi, \eta)^* \Psi(\xi, \eta) d\rho_n(\xi, \xi^*) d\rho_n(\eta, \eta^*) \quad \text{A.4.7}$$

It is then sufficient to use Lemma A.4.1

Remark A.4.3

1) In Lemma A.4.3 (α'^*, β) is the linear form on $E \times E$ defined by

$$\langle (\alpha'^*, \beta), \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} \rangle = \langle \alpha'^*, \alpha \rangle + \langle \beta, \beta' \rangle \quad \text{A.4.8}$$

2) We also have the formula

$$\begin{bmatrix} \mathbb{1} & -\lambda \\ -\lambda'^* & \mathbb{1} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbb{1} - \lambda \lambda'^*)^{-1} & (\mathbb{1} - \lambda \lambda'^*)^{-1} \lambda \\ (\mathbb{1} - \lambda'^* \lambda)^{-1} \lambda'^* & (\mathbb{1} - \lambda'^* \lambda)^{-1} \end{bmatrix} \quad \text{A.4.9}$$

APPENDIX 5

A CLASS OF BANACH SPACE OF KERNELS

In this Appendix we use the notations of section II.1

Definition A.5.1 :

Let $\alpha \in \mathbb{R}_+$, $M \in \mathbb{R}$, $m \in \mathbb{N}$. Let E be a finite dimensional Banach spaces. Then $\mathcal{N}_E(m, \alpha, M)$ is the Banach space of kernels from $\mathbb{R}^4 \times \mathbb{R}^4$ to E , normed by :

$$\|k\|_{\alpha, m, M} = \sum_{\substack{|p|, |q| \\ \leq m}} \sup_{\mathbb{R}^4 \times \mathbb{R}^4} \left\{ \frac{[1+(p^\circ q^\circ)]^\alpha [1+(\tilde{p}^\circ \tilde{q}^\circ)]^\alpha}{(1+|p|^2)^M (1+|q|^2)^M} \|\partial_p^\alpha \partial_q^\alpha k(p, q)\|_E \right\} \quad \text{A 5.1}$$

Here for $p \in \mathbb{R}^4$ we denote $p = (p^\circ, \tilde{p})$; $p^\circ \in \mathbb{R}$, $\tilde{p} \in \mathbb{R}^3$ and

$$|p| = (p^{\circ 2} + \tilde{p}^2)^{1/2} \quad \text{A 5.2}$$

Remark A.5.2 : The space $\mathcal{N}_E(m, \alpha, M)$ is increasing with M and decreasing with α and m .

Let $S_{\sigma, \alpha}(\lambda) = P(\partial \partial) \Delta_{\sigma, \alpha}(\lambda)$ where P is a polynomial of degree σ over $\mathbb{R}^4$ with values in $\mathcal{L}(E)$. (cf. Eq. II.1.10-11)

Let $(A_\mu)_{\mu=0,1,2,3}$ and v be functions in $\mathcal{S}(\mathbb{R}^4)$. We will put :

$$\hat{A}(p, q) = \tilde{v}(p-q) + \hat{A}_\mu \hat{A}^\mu(p-q) + \tilde{v} \hat{A}_\mu(p-q) (p^\mu + q^\mu / 2) \quad \text{A.5.3}$$

$$\hat{A}(p, q) = \gamma_\mu \hat{A}^\mu(p-q) + \tilde{v}(p-q) \quad \text{b)}$$

where $\hat{A}$ denotes Fourier transform, and γ_μ denotes the Dirac matrices.

Lemma A.5.3 : The mapping $(v, A_\mu) \rightarrow \tilde{A}$ (resp $(v, A_\mu) \rightarrow A$) from $\mathcal{S}(\mathbb{R}^4) \times \mathcal{S}'$ to $\mathcal{N}(m, \alpha, 1/4)$ (resp to $\mathcal{N}(m, \alpha, 0)$) is continuous.

Proof : Since $\tilde{v}$ and $\tilde{A}_\mu$ belongs to $\mathcal{S}(\mathbb{R}^4)$ [25] we can find for given α and m , a constant C such that for $|p| \leq m$ and $|q| \leq m$:

$$|\frac{\partial^\mu}{(\partial p)^\mu} \frac{\partial^\nu}{(\partial q)^\nu} \tilde{A}(p, q)| \leq C \frac{\text{Inf}[(1+|q|^2)^{1/2} (1+|p|^2)^{1/2}]}{[1+(p-q)^2]^\alpha [1+(\vec{p}-\vec{q})^2]^\alpha} \quad \text{A.5.4}$$

noticing that in A.5.3 a) one can use decompositions of $p_\mu + q_\mu$ as $2 p_\mu + (q_\mu - p_\mu)$ or as $2 q_\mu + (p_\mu - q_\mu)$, so that we get at most a linear growth in $|p|$ or in $|q|$ due to the fast decrease in $p-q$.

Using the inequality

$$\text{Inf}(|a||b|) \leq \sqrt{|a||b|} \quad \text{A.5.5}$$

the proof of Lemma A.5.3 is complete.

Lemma A.5.4 : If $\alpha > 3/2$ and $K \in \mathcal{N}_E^0(m, \alpha, 1/4)$, then $\hat{K}_{++}$ and $\hat{K}_{--}$ are bounded operators, and $\hat{K}_{+-}$, $\hat{K}_{-+}$ are in the Hilbert-Schmidt class. Moreover the mapping

$$K \rightarrow (\hat{K}_{++}, \hat{K}_{--}, \hat{K}_{+-}, \hat{K}_{-+}) \quad \text{A.5.6}$$

from $\mathcal{N}_E^0(m, \alpha, 1/4)$ is linear and continuous

Proof : a) Let us first examine $\hat{K}_{+-}$:

$$\|\hat{K}_{+-}\|_{H.S.} = C^{te} \int |K(p, -q)|^2 d\Omega_m(p) d\Omega_m(q)$$

$$\leq C^{te} \|K\|_{m, \alpha, 1/4}^2 \int \frac{d^3 \vec{p} d^3 \vec{q}}{[1+(\omega_p + \omega_q)^2]^\alpha [1+(\vec{p} - \vec{q})^2]^\alpha}$$

A.5.7

$$\leq C^{te} \|K\|_{m, \alpha, 1/4}^2 \left(\int \frac{d^3 \vec{p}}{(1+\vec{p}^2)^\alpha} \right) \left(\int \frac{d^3 \vec{q}}{(1+(\vec{p}-\vec{q})^2)^\alpha} \right)$$

Both integrals are equal and converge for $\alpha > 3/2$

b) By a theorem from Dunford and Schwarz [37], one has the following : Let (S, Σ, μ) be a positive measured space, and R a measurable function on $S \times S$ such that

$$\operatorname{ess\,sup}_t \int |R(s,t)| d\mu(s) \leq M \quad \operatorname{ess\,sup}_s \int |R(s,t)| d\mu(t) \leq M \quad A.5.8$$

Then the operator T defined by

$$g = Tf$$

$$g(s) = \int R(s,t) f(t) d\mu(t) \quad A.5.9$$

is a bounded operator in $L^2(S, \Sigma, \mu)$, and we have

$$\|T\| \leq M \quad A.5.10$$

This result generalizes L^2 -space inequalities for convolutions.

Here we choose :

$$S = \mathbb{R}^3 \quad d\mu(\vec{p}) = \frac{d^3\vec{p}}{2\omega_p} \quad R = \hat{K}_{++}$$

$$\int_{\vec{q}=\omega_p} |\hat{K}_{++}(p,q)| \frac{d^3q}{2\omega_q} \leq C^k \|K\|_{m,\alpha,1/4} (1+|p|^2)^{1/4} \int (1+(\vec{p}-\vec{q})^2)^{-\alpha} (2\omega_q)^{-1/2} d^3\vec{q} \quad A.5.11$$

if $\alpha > 3/2$, Eq. A.5.20 gives

$$\operatorname{ess\,sup} \int |\hat{K}_{++}(p,q)| d\Omega_m(q) \leq C^k \|K\|_{m,\alpha,1/4} \operatorname{ess\,sup} \frac{(1+|p|^2)^{1/4}}{(1+|\vec{p}|^2)^{1/4}} \leq C^k \|K\|_{m,\alpha,1/4} \quad A.5.12$$

since on mass-shell

$$1+|p|^2 = 2(1+\vec{p}^2) \quad A.5.13$$

The proof is complete.

Lemma A.5.5 : if $\alpha - M > 3/2$ and $K \in N(m, \alpha, M + 1/4)$ then $\hat{K}_{+-}$ and $\hat{K}_{-+}$ are in the Hilbert-Schmidt class, and there exists a dense domain $\mathcal{D}_+$ in $\mathcal{H}_+$ (resp $\mathcal{D}_-$ in $\mathcal{H}_-$) such that $\hat{K}_{++}$ (resp $\hat{K}_{--}$) is defined on $\mathcal{D}_+$ (resp $\mathcal{D}_-$) with

$$\hat{K}_{++}\mathcal{D}_+ \subset \mathcal{D}_+ \quad (\text{resp. } \hat{K}_{--}\mathcal{D}_- \subset \mathcal{D}_-) \quad \text{A.5.14}$$

Proof: The proof is the same as before. The domain $\mathcal{D}$ can be chosen as the space of $\mathcal{C}^m$ functions of fast decrease on the mass-shell.

Lemma A.5.6 : Let $m \geq 1$ be an integer, α, β be real positive numbers, M, N real numbers such that $M + N + \frac{\alpha-1}{2} \geq 0$. Then, there exists a constant C and indices γ and P such that

$$\|KS_{ex}L\|_{m, \gamma, P} \leq C \|K\|_{m, \alpha, N} \|L\|_{m, \beta, N} \quad \text{A.5.15}$$

If there exist $A \geq 0, B \geq 0$ with

$$A+B = M+N + \frac{\alpha-1}{2}, \quad \sup(\alpha-A, \beta-B) > 3/2 \quad \inf(\alpha-A, \beta-B) \geq 0 \quad \text{A.5.16}$$

Then

$$\gamma \leq \inf(\alpha-A, \beta-B); \quad P \geq \sup(M+A, N+B) \quad \text{A.5.17}$$

Proof : a) We need some auxiliary results

i) The following inequalities hold : $\forall x \in \mathbb{R}^n, \forall y \in \mathbb{R}^n$

$$1 + \|x\|^2 + \|y\|^2 \leq (1 + \|x\|^2)(1 + \|y\|^2) \quad \text{a)}$$

$$(1 + \|x+y\|^2) \leq 2(1 + \|x\|^2)(1 + \|y\|^2) \quad \text{b)} \quad \text{A.5.18}$$

$$\text{if } \|x\| \geq 1 \quad \frac{1}{\|x\|} \leq \frac{\sqrt{2}}{(1 + \|x\|^2)^{1/2}} \quad \text{c)}$$

ii) If x, y, z are in $\mathbb{R}^n$, and $\theta \geq 0$ the following inequality holds [38]

$$(1 + \|x - y\|^2)^{-\theta} (1 + \|y - z\|^2)^{-\theta} \leq C_\theta (1 + \|x - z\|^2)^{-\theta} \left\{ (1 + \|x - y\|^2)^{-\theta} + (1 + \|y - z\|^2)^{-\theta} \right\} \quad \text{A.5.19}$$

Indeed, $\forall \theta, \exists K_\theta$ such that

$$(|a| + |b|)^\theta \leq K_\theta (|a|^\theta + |b|^\theta) \quad \forall a, \forall b \in \mathbb{R}$$

And by

$$(1 + \|x - y\|^2) \leq 1 + 2(\|x - y\|^2 + \|y - z\|^2) \leq 2[(1 + \|x - y\|^2) + (1 + \|y - z\|^2)]$$

we obtain easily A.5.19.

iii) If x and y are in $\mathbb{R}^n$, $\alpha \geq 0$, $\beta \geq 0$ then if $\sup(\alpha, \beta) > \frac{n}{2}$

$$\int \frac{d^n y}{(1 + \|x - y\|^2)^\alpha (1 + \|y - z\|^2)^\beta} \leq \frac{C}{(1 + \|x - z\|^2)^{\inf(\alpha, \beta)}} \quad \text{A.5.20}$$

as can be seen by application of A.5.19 and of the Holder inequality.

b) We can without loss of generality, choose the particle mass m to be unity

i) Up to a multiplicative constant, we have in momentum space

$$\widetilde{K S_+ L}(p, q) = c^* e \int_{\vec{r} \in \omega_r} \frac{\tilde{K}(p, r) \tilde{P}(r) \tilde{L}(r, q) d^3 \vec{r}}{(1 + \vec{r}^2)^{1/2}} \quad \omega_r = (1 + \vec{r}^2)^{1/2}$$

This expression can be majorised by

$$\left\| \frac{\partial^N}{\partial \vec{p}^N} \frac{\partial^M}{\partial \vec{q}^M} K S_+ L \right\| \leq c^* e \quad \|K\|_{m, \alpha, H} \|N\|_{m, \beta, N} (1 + |p|^2)^H (1 + |q|^2)^N \dots$$

$$\dots \int \frac{(1 + |r|^2)^{H+N+\frac{\sigma-1}{2}} d^3 \vec{r}}{(1 + (p - \omega_r)^2)^\alpha (1 + (q - \omega_r)^2)^\alpha (1 + (\vec{p} - \vec{r})^2)^\alpha (1 + (\vec{r} - \vec{q})^2)^\alpha} \quad \text{A.5.21}$$

Because $r^0 = \omega_r \implies 1 + |r|^2 = 2(1 + \tilde{r}^2)$

Let us apply A.5.19, and remark that A.5.18 b)

$$\frac{1 + \tilde{r}^2}{1 + (\tilde{p} - \tilde{r})^2} \leq 2(1 + \tilde{p}^2) \quad \text{A.5.22}$$

Then, if $A \geq 0$, $B \geq 0$ satisfy A.5.16, and γ and P are defined by A.5.17 we find, with A.5.20 the result for $S_{ev} = S_+$

ii) Let us prove the lemma for $S_{ev} = S_r$.

Up to a constant, we have

$$\tilde{S}_r(r) = \frac{P(r)}{(r^0 - \tilde{u}_0)^2 - \omega_r^2} = \frac{P(r)}{\omega_r} \left[\frac{1}{r^0 - \tilde{u}_0 - \omega_r} + \frac{1}{r^0 - \tilde{u}_0 + \omega_r} \right] \quad \text{A.5.23}$$

Thus, $KS_r L$ can be split into two parts which can both be majorised in similar ways.

For instance

$$\int \frac{\tilde{K}(p,r) P(r) \tilde{L}(r,q) d^4 r}{2\omega_r (r^0 - \tilde{u}_0 - \omega_r)} = \int_{|r^0 - \omega| \geq 1} \dots + \int_{|r^0 - \omega| \leq 1} \dots \quad \text{A.5.24}$$

We can immediatly majorise the second term of A.5.24

$$\int_{\substack{|r^0 - \omega| \leq 1 \\ |r^2 - \omega|^2 \leq 1}} \dots = \int_{\substack{|r^2 - \omega|^2 \leq 1 \\ |r^0 - \omega| \leq 1}} \frac{\tilde{K} \tilde{P} L(p,q,r) - \tilde{K} \tilde{P} L(p,q,r)|_{r^0 = \omega_r}}{r^0 - \omega - \tilde{u}_0} + \left(\frac{d^3 r}{2\omega_r} \tilde{K} \tilde{P} L|_{p,q} \right) \int_{\substack{r^0 = \omega_r \\ \omega - 1}}^{\omega + 1} \frac{dr_0}{r^0 - \omega - \tilde{u}_0} \quad \text{A.5.25}$$

But

$$\int_{-1}^{+1} \frac{dx}{x - \tilde{u}_0} = -\tilde{u}_0 \pi \quad \text{A.5.26}$$

and the proof i) for $S_{ex} = S_+$ is sufficient to obtain the good majorisation for the second term of A.5.25

The first term of A.5.25 is majorised in the same way because

$$\left\| \frac{\widetilde{K\tilde{P}L}(p,q,r)}{r^2 - \omega - \varepsilon_0} - \widetilde{K\tilde{P}L}|_{r^2 = \omega, r} \right\| \leq \sup_{|r^2 - \omega| \leq 1} \left\| \frac{\partial}{\partial r_0} \widetilde{K\tilde{P}L}(p,q,r) \right\| \quad \text{A.5.27}$$

It is at this point that we need the derivability of $\widetilde{K}$ and $\widetilde{L}$ ($m \geq 1$)
If we remark that

$$(1 + |r|^2) \leq e^{te} (1 + \tilde{r}^2) \quad \text{if } |r^2 - \omega| \leq 1 \quad \text{A.5.28}$$

and $\int_{|r^2 - \omega| \leq 1} dr^0 = 2$ we find the same majorisation as in i).

iii) Now we have to majorise $\int_{|r^2 - \omega| \geq 1} \dots$ in A.5.24

$$\left\| \frac{\partial^M}{\partial p^M} \frac{\partial^N}{\partial q^N} \int_{|r^2 - \omega| \geq 1} \dots \right\| \leq e^{te} \|K\|_{m, \alpha, M} \|L\|_{m, \beta, N} (1 + |p|^2)^M (1 + |q|^2)^N \dots$$

$$\dots \int \frac{(1 + |r|^2)^{M+N+\frac{\alpha-1}{2}} (1 + |\vec{p}-\vec{r}|^2)^{-A} (1 + |\vec{q}-\vec{r}|^2)^{-B}}{(1 + (p^0 - r^0)^2)^{\alpha-A} (1 + (\vec{p}-\vec{r})^2)^{\alpha-A} (1 + (q^0 - r^0)^2)^{\beta-B} (1 + (\vec{q}-\vec{r})^2)^{\beta-B}} \quad \text{A.5.29}$$

In this expression we have used for $|r^2 - \omega| \geq 1$ (by A.5.18 c) and b))

$$\frac{(1 + |r|^2)^{1/2}}{(1 + \tilde{r}^2) |r^2 - \omega|} \leq \frac{\sqrt{2} (1 + |r|^2)^{1/2}}{(1 + \tilde{r}^2) (1 + (r^0 - \omega)^2)^{1/2}}$$

$$\leq \frac{e^{te} (1 + |r|^2)^{1/2}}{(1 + \tilde{r}^2)^{1/2} (1 + (\tilde{r} - \omega)^2)^{1/2}} \leq e^{te} \quad \text{A.5.30}$$

Using again A.5.22, A.5.20 we find the expected result.

iv) If S_{ex} is any other two point function, it can be expressed in term of S_r and S_+ . Therefore the lemma is shown.

Corollary A.5.7: Let $m \geq 1$ be an integer, α, β be real positive numbers with $\alpha \leq \beta$. Then :

$$\|K \Delta_{ex} L\|_{m, \alpha, \ell_4} \leq C^{te} \|K\|_{m, \alpha, \ell_4} \|L\|_{m, \beta, \ell_4} \quad a)$$

A.5.31

$$\|K S_{ex} L\|_{m, \alpha, 0} \leq C^{te} \|K\|_{m, \alpha, 0} \|L\|_{m, \beta, 0} \quad b)$$

where

$$S_{ex} = (-i \gamma_\mu \partial_\mu + m) \Delta_{ex}$$

Fig. I

$$I(x,y) = \begin{array}{c} \text{diagram with 1 blob} \\ + \\ \text{diagram with 2 blobs} \\ + \dots + \\ \underbrace{\text{diagram with } n \text{ blobs}}_n + \dots \end{array}$$

Fig. II

$$G = \begin{array}{c} \text{diagram with 1 blob} \\ = \sum_{n \geq 0} \text{diagram with } n \text{ blobs} \\ = \text{diagram with 0 blobs} + \text{diagram with 1 blob} \end{array}$$

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QUANTIZED FIELDS IN EXTERNAL FIELD

II. EXISTENCE THEOREMS

by

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ABSTRACT : This is the second part of an article devoted to the study of quantized fields interacting with a smooth classical external field with fast space time decrease. The case of a charged scalar field is considered first. The existence of the corresponding Green's functions is proved. For weak fields, as well as pure electric or scalar external fields, the Bogolioubov S-operator defined in part I of this work is shown to be unitary, covariant, causal up-to-a-phase. Its perturbation expansion is shown to converge on a dense set in Fock space. These results are generalised to a class of higher spin quantized fields, "nicely" coupled to external fields, which includes the Dirac theory, and, in the case of minimal and magnetic dipole coupling, the Petiau-Duffin-Kemmer theory.

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I - INTRODUCTION

In the first part [1] of this paper, we have studied the general properties of the Green functions and of the Bogolioubov S-operator, for the problem of a spin zero quantized field interacting with a classical external field.

Most of these properties can be extended to quantized fields with higher spins. In fact there is no difference between zero spin and arbitrary integer spins. On the other hand, in the case of half integer spins, the properties of the Green functions undergo some well-known algebraic modifications. Besides, the method used in [1] to construct the S-operator, cannot be used in the case of fermion fields, which however could be treated by other methods [2].

The main result of the first part was that the S-operator, constructed according to the perturbation scheme, is covariant, unitary, and causal up-to-a phase if and only if there exists a Non- Perturbative solution (N-P in [1] def. II.2.3) of the following integral equation :

$$I = A + A \Delta_F I = A + I \Delta_F A \quad \text{I.1}$$

where $A(x,y) = [\psi(x) + A_\mu A^\mu(x)] \delta(x-y) + i [A_\mu(x) + A_\mu(y)] \partial^\mu \delta(x-y)$. ψ , A_μ denoting respectively the scalar and the vector part of the external field, and Δ_F the usual free Feynman propagator.

In the present paper we shall establish the existence of such solutions for some classes of external fields.

In the case of a spin zero quantized field, the main result is the following (see corollary II.2.5) :

Theorem I.1 :

Let ψ , A_μ belong to the space $\mathcal{C}_0^\infty(\mathbb{R}^4)$ of the functions with compact support in $\mathbb{R}^4$ [3], then, the equation :

$$I(\lambda) = A(\lambda) + A(\lambda) \Delta_F I(\lambda) = A(\lambda) + I(\lambda) \Delta_F A(\lambda) \quad \text{I.2}$$

where $A(\lambda)$ corresponds to $\lambda \psi$, λA_μ ($\lambda \in \mathbb{C}$), has a unique N-P solution if λ is small enough; this solution is analytic for in the N-P norm topology, for some $\epsilon > 0$.

The N-P norm topology will be defined in the first section (cf. formula II.1.4) The previous theorem can be generalized in various directions. In particular it also holds, if we replace N.P. by W.N.P. (cf. [1] def. II.2.2) for any λ in the complement of a discrete subset Σ of $\mathbb{C}$. Moreover if v and A_μ are real, Σ does not intersect the real axis.

Another extension can be made using a result proved by Schroer, Seiler, and Swieca [4]: there exists an N-P solution, for any real λ , when v and A_0 are real and $A_i = 0$ ($i=1,2,3$).

These results yield Theorem II.3.2 according to which the S-operator constructed in part I, is strongly continuous with respect to the external field, in a neighbourhood of zero, and expandable into a strongly convergent power series on a dense domain in Fock space.

We shall also extend these results to quantized fields with higher spins, if the interaction kernel A satisfies some special properties:

Let ψ be a charged Wightmann free field defined as the solution of the first order covariant partial differential system.

$$\psi \left(\left(\frac{1}{i} \beta_\mu \partial^\mu - m \right) f \right) = 0 \quad \forall f \in \mathcal{S} \quad \text{I.3}$$

where β_μ are $N \times N$ matrices and $m > 0$.

Definition I.2 :

Let Φ be an $N \times N$ matrix valued $\mathcal{C}_0^\infty(\mathbb{R}^4)$ function. Φ will be called a nice interaction, if one can find four $N \times N$ matrix valued partial differential operators A_R, A_L, B_R, B_L satisfying the following conditions:

i) The coefficients of A_R, A_L, B_R, B_L are polynomials in Φ and its derivatives.

ii) A_R and A_L are partial differential operators of the first order.

iii) If $d(\frac{1}{i}\partial)$ is the Klein-Gordon divisor [5,6] of $(\frac{1}{i}\beta_\mu \partial^\mu - m)$ then:

$$(d(-i\partial) + B_L)(-i\beta_\mu \partial^\mu - m - \Phi) = -(\square + m^2)\mathbb{1} + A_L$$

$$(-i\beta_\mu \partial^\mu - m - \Phi)(d(-i\partial) + B_R) = -(\square + m^2)\mathbb{1} + A_R$$

I.4

Non trivial interactions of this type will be shown to exist. In particular all $\mathcal{C}_0^\infty$ interactions are nice if the β_μ 's are the Dirac matrices; the minimal and dipole couplings occuring in the Petiau-Duffin-Kemmer equations [7] are nice interactions.

Because of equation I.4 and classical theorems about the Cauchy problem for the Klein-Gordon equations, with interactions, there exist advanced and retarded elementary solutions for the coupled differential operator $(-i\beta_\mu \partial^\mu - m - \Phi)$ with support in $\overline{V}^\pm$. Therefore nice interactions do not lead to acausality [8]. Moreover we will show the following result :

Theorem I.3

Let Φ be a nice interaction. Then the equation

$$I(\lambda) = \Phi(\lambda) + \Phi(\lambda) S_F I(\lambda) = \Phi(\lambda) + I(\lambda) S_F \Phi(\lambda) \quad \text{a) I.5}$$

$$\text{with} \quad \Phi(\lambda) = \lambda \Phi(x) \delta(x-y) \quad (\lambda \in \mathbb{C}) \quad \text{b)}$$

$$S_F = d(-i\partial) \Delta_F \quad \text{c)}$$

has a unique N-P solution if λ is small enough and Φ is a hermitian matrix-valued function.

Therefore the quantized theory can be made in these cases.

These properties solve the problem posed by Wightman [6] about the existence of a retarded solution and the absence of acausality [8], in the case of this special kind of interactions.

Some of the previous results are either stated or announced in [6] and sharpened here with the help of the stronger assumptions made on the external field. It seems also that some convergence properties of the perturbation series for weak external fields have been known for some time by L. Garding.

This article is divided as follows :

Section II deals with the case of spin zero quantized fields.

Section III generalized the results of section II to higher spin quantized fields coupled to nice external field.

An appendix is devoted to a summary of some relevant results on a priory bound for hyperbolic partial differential operators.

II. EXISTENCE THEOREMS FOR SPIN ZERO QUANTIZED FIELDS

II.1 Weak external fields

In our preceding paper [1] we have introduced the family $\mathcal{N}_{E_N}(m, \alpha, M)$ of Banach spaces, whose elements are $\mathcal{C}^m$ functions defined in $\mathbb{R}^4 \times \mathbb{R}^4$ with values in a finite dimensional Banach space E_N , and normed by :

$$\|K\|_{m, \alpha, M} = \sum_{|\mu|, |\nu| \leq m} \sup_{(p, q) \in \mathbb{R}^8} \left\{ \frac{[1 + (p^0 - q^0)^2]^\alpha [1 + (p^0 + q^0)^2]^\alpha}{[1 + |p|^2]^M [1 + |q|^2]^M} \|\partial_p^\mu \partial_q^\nu K(p, q)\|_{E_N} \right\} \quad \text{II.1.1}$$

Let us recall that the Fourier transform $\tilde{A}$ of A :

$$\tilde{A}(p, q) = \tilde{v}(p - q) + \tilde{A}_\mu \tilde{A}^\mu(p - q) + \tilde{v} \tilde{A}_\mu(p - q)(p^\mu + q^\mu) \quad \text{II.1.2}$$

is in $\mathcal{N}_d(m, \alpha, 1/4)$ $\forall m \geq 0 \quad \forall \alpha \geq 0$. ([1] Appendix 5).
if v, A_μ are in $\mathcal{S}(\mathbb{R}^4)$ [9] .

We have also associated to a kernel K the operator $\hat{K}$ on the one particle space $\mathcal{H}_+ \oplus \mathcal{H}_-$ of our theory defined by restricting the Fourier transform $\tilde{K}$ on the positive or negative mass-shell. We will denote by $\mathfrak{S}_{N, \mathbb{P}}$ the supspace in $\mathcal{L}(\mathcal{H}_+ \oplus \mathcal{H}_-)$ of operators of the type

$$\hat{K} = \begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix} \quad \text{II.1.3}$$

such that K_{+-} and K_{-+} are in the Hilbert Schmidt class [10] . We shall define in $\mathfrak{S}_{N, \mathbb{P}}$ the N-P norm by :

$$\|\hat{K}\|_{N, \mathbb{P}} = \text{Max} \left\{ \|K_{++}\|_{op}, \|K_{--}\|_{op}, \|K_{+-}\|_{H.S.}, \|K_{-+}\|_{H.S.} \right\} \quad \text{II.1.4}$$

The following results are shown in [1] :

- (K1) If $m \geq 1$, $\beta \geq \alpha$, $\beta > 3/2$, then there exists a constant $C > 0$ such that :

$$\|K \tilde{\Delta}_{ex} L\|_{m, \alpha, 1/4} \leq C \|K\|_{m, \alpha, 1/4} \|L\|_{m, \beta, 1/4} \quad \text{II.1.5}$$

where Δ_{ex} is any of the two point functions of the free field.

([1] : Lemma A.5.6).

(K2) The mapping $K \rightarrow \hat{K}$ is linear and bounded from $N(m, \alpha, 1/4)$ to $\mathcal{G}_{N,D}$ if $\alpha > 3/2$. ([1] Lemma A.5.4).

Therefore, we are able to show the first main theorem :

Theorem II.1.1 :

Let v, A_μ be in $\mathcal{S}(\mathbb{R}^4)$ and λ be a complex number. Then the equation :

$$K(\lambda) = A(\lambda) + A(\lambda) \Delta_{ex} K(\lambda) = A(\lambda) + K(\lambda) \Delta_{ex} A(\lambda) \quad \text{II.1.6}$$

(where $A(\lambda)$ is obtained from equation I.1.b by multiplying v, A_μ by λ) has a unique solution with $\tilde{K} \in N(m, \alpha, 1/4)$ for $\alpha \geq 0$ and $m \geq 0$, which is analytic with respect to λ in a neighbourhood of zero.

Remark II.1.2

This theorem also applies if v and A_μ are matrix-valued functions.

Proof : Let $m \geq 1$ and $\beta \geq \alpha, \beta > 3/2$. Using the result K1) one finds :

$$\|\tilde{A}(\lambda) \Delta_{ex} \dots \tilde{A}(\lambda) \Delta_{ex} \tilde{A}(\lambda)\|_{m, \alpha, 1/4} \leq C^n \|\tilde{A}(\lambda)\|_{m, \beta, 1/4}^n \quad \text{II.1.7}$$

Because (cf. [1] Prop. II.2.4),

is finite.

Therefore the Neumann series:

$$K(\lambda) = \sum_{n \geq 0} A(\lambda) [\Delta_{ex} A(\lambda)]^n \quad \text{II.1.8}$$

converges in $N(m, \alpha, 1/4)$, if λ is small enough since $A(\lambda) = O(\lambda)$. Since $A(\lambda)$ is a polynomial in λ , $K(\lambda)$ is analytic for small λ .

Corollary II.1.3

The solution $K(\lambda)$ of II.1.6 defines an analytic family $\hat{K}(\lambda)$ of elements of $\mathcal{G}_{N,P}$ for λ small enough.

Proof : This can be seen by using K2).

II.2 Strong external fields.

Let us denote, as in ref. [1] the kernel $K(\lambda)$ defined by equation II.1.6 by $J_r(\lambda)$ (resp. $J_a(\lambda)$, $I(\lambda)$, $\bar{I}(\lambda)$) if $\Delta_{ex} = \Delta_r$ (resp. Δ_a , Δ_F , $\Delta_{\bar{F}}$) and let $\hat{K}(\lambda) = \Gamma J_R(\lambda) \Gamma^{-1}$ (resp. $\Gamma J_A(\lambda) \Gamma^{-1}$, $I_S(\lambda)$, $\bar{I}_S(\lambda)$) where

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{II.2.1}$$

The main result is then the following (see [1] def. II.2.2)

Theorem II.2.1 :

Let v, A_μ be in $\mathcal{C}_0^\infty[\mathbb{R}^4]$ [3]. Then there exists a unique W-N-P solution for $J_r(\lambda)$ which is analytic in λ in the whole complex plane in the following sense :

- i) $\lambda \rightarrow J_r(\lambda)$ is analytic if one chooses the simple convergence topology [30] of linear continuous operators from $\mathcal{O}_H(\mathbb{R}^4)$ to $\mathcal{Y}(\mathbb{R}^4)$ [9].
- ii) $\lambda \rightarrow J_R(\lambda)$ is an analytic family of bounded operators, such that $J_{+-}(\lambda)$ and $J_{-+}(\lambda)$ are compact [10]

Remark II.2.2

The same theorem holds for $J_a(\lambda)$.

We first need some partial results :

Lemma II.2.3 :

If $v, A_\mu \in \mathcal{C}_0^\infty(\mathbb{R}^4)$ and $\lambda_0 \in \mathbb{C}$, then $(1 - A(\lambda)\Delta_r)$ defines an analytic family of automorphisms of $\mathcal{S}(\mathbb{R}^4)$ for the simple convergence topology in a neighbourhood of λ_0 .

The proof of this lemma is the main content of the appendix. This result can be extended without change to the case in which v, A_μ are in $\mathcal{S}(\mathbb{R}^4)$ with support in $[t_0, t_1] \times \mathbb{R}^3$ for some $t_0, t_1 \in \mathbb{R}$.

Lemma II.2.4

If $v, A_\mu \in \mathcal{C}_0^\infty(\mathbb{R}^4)$ and $\lambda_0 \in \mathbb{C}$, then $\mathcal{T}_R(\lambda)$ defines an analytic family of bounded operators on $\mathcal{H}_+ \oplus \mathcal{H}_-$, in a neighbourhood of λ_0 .

The proof of this lemma is also in the appendix.

Lemma II.2.5

Let D be a domain in $\mathbb{C}$, $\mathcal{H}$ be a Hilbert space, and $\hat{K}(\lambda)$ an analytic family of bounded operator on $\mathcal{H}$. If there exists a subdomain D' in D , such that $\hat{K}(\lambda)$ is compact for any λ in D' , then $\hat{K}(\lambda)$ is compact for any λ in D .

Proof : Let λ_0 be a point of $D - D'$ such that $\hat{K}(\lambda_0)$ is not compact. Then by the Hahn-Banach theorem, there exists a linear form ρ on $\mathcal{L}(\mathcal{H})$ such that :

$$\rho(\hat{K}(\lambda_0)) = 1 \quad \rho(\hat{K}) = 0 \quad \text{if } \hat{K} \text{ is compact}$$

since compact operators span a closed subspace of $\mathcal{L}(\mathcal{H})$ [10].

Therefore $\lambda \rightarrow \rho(\hat{K}(\lambda))$ cannot be analytic in D [11] which contradicts the hypothesis.

Proof of theorem II.2.1 :

i) Because $\mathcal{T}_r(\lambda) = (1 - A(\lambda)\Delta_r)^{-1}A(\lambda)$ the first part of theorem II.2.1 follows from Lemma II.2.3.

ii) From corollary II.1.3 we know that $J_R(\lambda)$ is N-P for small λ . Therefore if λ is small, $J_{r+-}(\lambda)$ and $J_{r-+}(\lambda)$ are Hilbert-Schmidt and thus compact. From Lemma II.2.4 and II.2.5 and II.2.5, the other part of theorem II.2.1 follows.

II.3 Existence and properties of the Bogolioubov S-operator :

We are now able to prove theorem I.1 :

Theorem II.3.1 :

Let v and A_μ be in $\mathcal{C}_0^\infty(\mathbb{R}^4)$, and $\lambda \in \mathbb{C}$.

a) Then the equation

$$I(\lambda) = A(\lambda) + I(\lambda) \Delta_{ex} A(\lambda) = A(\lambda) + A(\lambda) \Delta_{ex} I(\lambda) \quad \text{II.3.1}$$

has a unique W-N-P solution except if λ belongs to some discrete subset Σ of $\mathbb{C}$.

b) $\lambda \rightarrow I(\lambda)$ is analytic in $\mathbb{C} - \Sigma$ for the simple convergence topology of linear continuous operators from $\mathcal{O}_H(\mathbb{R}^4)$ to $\mathcal{P}(\mathbb{R}^4)$.

c) $\lambda \rightarrow I_S(\lambda)$ is analytic in $\mathbb{C} - \Sigma$ for the topology of $\mathcal{L}(\mathcal{H}_+ \oplus \mathcal{H}_-)$.

d) $\exists \epsilon > 0$, such that $I(\lambda)$ is N-P and $\lambda \rightarrow I_S(\lambda)$ is analytic for the N-P norm topology, if $|\lambda| < \epsilon$.

e) If v, A_μ are real, Σ does not intersect the real line.

f) If v, A_μ are real and $A_i = 0$ ($i=1,2,3$) then $I_S(\lambda)$ is N-P for all real λ 's.

Proof : From theorem II.2.1 one knows the existence and unicity of $J_r(\lambda)$ as a W-N-P kernel. The equivalence theorem ([1] theorem II.4.1) gives the existence and unicity of $I(\lambda)$ if and only if, $\Pi_+ J_{r+-}(\lambda) J_{a-+}(\lambda)$ is invertible. Since $\lambda \rightarrow J_{r+-}(\lambda)$ (resp. $\lambda \rightarrow J_{a-+}(\lambda)$) is an analytic family of compact operators, then, for all λ by a slight extension of the classical Fredholm alternative [10,11,12] there exists a discrete subset Σ of $\mathbb{C}$ such that $\Pi_+ J_{r+-}(\lambda) J_{a-+}(\lambda)$ is invertible in $\mathbb{C} - \Sigma$. This proves a) b), c)

- d) follows from theorem II.1.1
- e) follows from ([1], theorem II.4.1)
- f) follows from a theorem given by Schrper-Seiler-Swieca [4].

Using the results given in ([1] chapter III) about the construction of a Bogolioubov S-Operator up-to-a-phase one finds

Theorem II.3.2

Let v, A_μ be real functions in $\mathcal{C}_0^\infty(\mathbb{R}^4)$ and let $S_0(\lambda v, \lambda A_\mu)$ be the quantized S-operator [1]:

$$S_0(\lambda v, \lambda A_\mu) = \det(\mathbb{1}_+ - I_{\pm} \bar{I}_{\pm}(\lambda))^{+1/2} : e^{i \int \varphi^* I(\lambda) \varphi} : \quad \text{II.3.2}$$

Then, if $\Phi_{f,G}$ is the coherent state with wave functions f, G :

- B0) $(\lambda, f, G) \rightarrow S_0(\lambda v, \lambda A_\mu) \Phi_{f,G}$ is analytic in the domain $|\lambda| < \epsilon$, $f \in \mathcal{H}_+$, $G \in \mathcal{H}'_-$, in the strong topology of Fock space, for some $\epsilon > 0$.

- B1) $(v, A_\mu) \rightarrow S_0(v, A_\mu)$ is strongly continuous in a neighbourhood of zero in the space of real valued $\mathcal{C}_0^\infty(\mathbb{R}^4)$ functions

- B2)

$$S_0(\lambda v, \lambda A_\mu) = \mathbb{1} + i \lambda \int : \varphi^* \varphi : v(x) + i \lambda \int : \varphi^* \overleftrightarrow{\partial}_\mu \varphi : A_\mu(x) + O(\lambda^2) \quad \text{II.3.3.}$$

on the dense domain spanned by vectors with finite number of particles.

- Moreover if $\lambda \in \mathbb{R}$ is small enough, or if $A_i = 0$ ($i=1,2,3$)

Then

- B3) $S_0(\lambda v, \lambda A_\mu)$ is relativistically covariant

- B4) $S_0(\lambda v, \lambda A_\mu)$ is unitary

- B5) $S_0(\lambda v, \lambda A_\mu)$ is causal up-to-a phase.

Proof : Properties B3, B4 and B5 follow from theorem II.4.1 of [1] , and the fact that $I(\lambda)$ is N-P under the hypothesis of the theorem .

Property Bo . One can write ([1] eq. III.3.1)

$$S_c(A_\nu, A_\mu) \Phi_{f,G} = \det(I_+ - I_{+-}(\lambda) \bar{I}_{-+}(\lambda))^{1/2} e^{i\langle G, \bar{I}_{-+}(\lambda) f \rangle} \dots e^{i\langle a^+ I_{+-}(\lambda) b \rangle} \Phi_{(I_+ + i I_{+-}(\lambda))f, G(I_+ + i I_{-+}(\lambda))} \quad \text{II.3.4}$$

Note in particular that

$$\overline{I_{-+}(\lambda)} = I_{+-}(\bar{\lambda})^* \quad \text{II.3.5}$$

Since ν and A_μ are real functions.

The analyticity of :

$$(k, f, G) \rightarrow e^{i\langle a^+ k b \rangle} \Phi_{f,G} \quad \text{in the domain } \|K\|_{H.S} < +\infty \quad \text{II.3.6}$$

and $\|K\|_{op} \leq 1, f \in \mathcal{H}_+, G \in \mathcal{H}'_-$

and of :

$$K \rightarrow \det(1-K)^{1/2} \quad \text{if } K \text{ is trace class with } \|K\|_{op} \leq 1$$

II.3.7

yield to Bo).

Property B2. It is a simple consequence of Bo) and of the fact that

$I(\lambda)$ is $O(1)$ as $\lambda \rightarrow 0$. In view of the analyticity with respect to f and G , one can generate vectors with a finite number of particles by taking derivatives with respect to f and G in equation II.3.4.

Property B1. Recall that ([1] Appendix 5) by theorem II.1.1 and by property K2), of section II.1 the map

$$(\nu, A_\mu) \rightarrow I_S \quad \text{II.3.8}$$

is continuous in the N-P norm topology at least in a neighbourhood of zero in $\mathcal{C}_0^\infty(\mathbb{R}^4)$. The analyticity property (eq. II.3.6) for generalized coherent states implies that :

$$(v, A_\mu) \rightarrow S_o(v, A_\mu) \Phi_{f, G} \quad \text{II.3.9}$$

is strongly continuous if v, A_μ are small enough.

If we now restrict ourselves to real valued functions then $S_o(v, A_\mu)$ is unitary. Therefore ,

$$(v, A_\mu) \rightarrow S_o(v, A_\mu) \quad \text{II.3.10}$$

is strongly continuous in a neighbourhood of zero in the subspace of real valued $\mathcal{C}_0^\infty(\mathbb{R}^4)$ functions.

III. HIGHER SPIN QUANTIZED FIELDS IN EXTERNAL FIELDS .

III.1. A special class of free fields

We now consider a charged Wightman [13, 14] free field $\psi(x)$ describing an irreducible system of mass $m > 0$, spin " s " and charge 1, transforming under the restricted Poincaré group as follows :

$$U(a, A) \psi_\alpha(x) U(a, A)^{-1} = \sum_{\beta} L_{\beta\alpha}(A^{-1}) \psi_\beta(Ax + a) \quad \text{III.1.1}$$

where L is some finite dimensional representation of the group $SL(2, \mathbb{C})$.

The classification of such fields was given by several authors [15a, 15b, 16], at least for fields without internal symmetries. If one introduces the $U(1)$ symmetry which specifies the charge, these results can be trivially extended. The main result is the following :

Proposition III.1.1 [16] A necessary and sufficient condition for the

Wightman charged field ψ , with transformation law given by equation III.1.1, to be an irreducible free field is that :

i) there exists $\lambda > 0$, and irreducible $SU(2)$ -invariant spin " s " orthogonal projector P_0 in the space E_λ of the representation L

ii) the non zero two point Wightman functions are :

$$(\Omega, \psi(f)^* \psi(g) \Omega) = \lambda \int d\Omega_m(p) \langle \tilde{f}(p), L[p]^*{}^{-1} P_0 L[p]^{-1} \tilde{g}(p) \rangle$$

$$(\Omega, \psi(g) \psi(f)^* \Omega) = \lambda \int d\Omega_m(p) \langle \tilde{f}(-p), L[p]^*{}^{-1} R P_0 R L[p]^{-1} \tilde{g}(-p) \rangle$$

III.1.2

- Here $d\Omega_m$ is the Lorentz invariant measure of the positive mass-shell, $[p]$ is a solution of

$$[p][p]^* = \frac{p^0 + \vec{p}\vec{\sigma}}{m} \quad p^0 = (m^2 + \vec{p}^2)^{1/2} \quad \text{III.1.3}$$

$$\mathcal{R} = \mathcal{L}(\mathbb{1}, -\mathbb{1})$$

III.1.4

if $\mathcal{L}(A, B)$ is the analytic continuation of $\mathcal{L}$ to $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ [14, 16, 17]

$\tilde{f}$ is the Fourier transform of the function f , which will be assumed to belong to $\mathcal{S}(\mathbb{R}^4) \otimes E_N$.

We shall restrict ourselves to a class of fields defined by a first order system of partial differential equations [5, 6]. Recalling that [16] it is always possible to choose an inner product $\langle, \rangle$ in E_N such that

$$\mathcal{L}(A)^* = \mathcal{L}(A^*)$$

III.1.5

we shall assume the following axioms ([5]) :

S1) There exists a linear mapping $p \mapsto \beta(p)$ from $\mathbb{R}^4$ to $\mathcal{L}(E_N)$ which is covariant

$$\mathcal{L}(A) \beta(p) \mathcal{L}(A^{-1}) = \beta(Ap) \quad \forall A \in SL(2, \mathbb{C}) \quad \forall p \in \mathbb{R}^4$$

S2) There exists an invertible element η of $\mathcal{L}(E_N)$ such that :

$$\begin{aligned} \eta \mathcal{L}(A) \eta^{-1} &= \mathcal{L}(A^{*-1}) & \text{a)} \\ \eta \beta(p) \eta^{-1} &= \beta(p)^* & \text{b)} \end{aligned} \quad \text{III.1.7}$$

η can be chosen [18, 19] up to a sign such that,

$$\eta = \eta^* = \eta^{-1} \quad \text{III.1.8}$$

S3) Let $T(p)$ be an $\mathcal{L}(E_N)$ -valued tempered distribution such that

$$\begin{aligned} \mathcal{L}(A) T(p) \mathcal{L}(A)^{-1} &= T(Ap) & \text{a)} \\ T(p) (\beta(p) - m) &= 0 = (\beta(p) - m) T(p) & \text{b)} \end{aligned} \quad \text{III.1.9}$$

Then $\eta T(p)$ is a linear combination of the Fourier transform of the two-point Wightman functions of the irreducible Wightman field ψ .

Before characterizing this class of fields, let us remark that axioms S1 and S2 restrict the choice of the representation L [18, 19, 20]. However, for an arbitrary spin "s" it is possible to construct such a field [21, 22]. For instance, the Dirac field and the Petiau-Duffin-Kemmer field are in this class.

Now, we have the following preliminary result :

Lemma II.1.2 :

Let ψ be a charged Wightman field satisfying equation III.1.1, and axioms S1, S2, S3. Then ψ is a free field with non negative mass.

Proof : Let $\tilde{W}(p)$ be the Fourier transform of a non zero two-point Wightman function for ψ . By S3) it is solution of

$$\tilde{W}(p)(\beta p - m) = 0 \quad \text{III.1.10}$$

By a classical theorem on distributions [3, 9], the support of $\tilde{W}(p)$ is contained in the subset σ of $\mathbb{R}^4$ in which $\beta(p) - m$ is non invertible.

Because E_N is finite dimensional and $\beta(p)$ is covariant by S1), σ is a finite union of hyperboloids (with one or two shells). In view of S3), the only possible solutions of III.1.10 must describe an irreducible Wightman field. Therefore, σ is reduced to a two-shell hyperboloid, since zero mass as well as imaginary mass solutions are excluded from the Wightman theory, and the decomposability into several hyperboloids is excluded by the irreducibility hypothesis. Therefore

$\exists m' > 0$ with

$$\sigma = \{ p \in \mathbb{R}^4; p^2 = m'^2 \} \quad \text{III.1.11}$$

Now [13, 14],

i) $\text{supp } \tilde{W} = \sigma \cap V^+$ by covariance and spectral condition.

ii) $\tilde{W}(p)$ is a covariant positive measure, and therefore it satisfies the Klein-Gordon equation

$$(p^2 - m'^2) \tilde{W}(p) = 0 \quad \text{III.1.12}$$

iii) ψ is a generalized free field by the Borchers- Greenberg theorem [23]

iv) ii) and iii) prove that ψ is a solution of the Klein-Gordon equation corresponding to mass m' .

Lemma III.1.3 :

Let β be a linear mapping from $\mathbb{R}^4$ to E_N satisfying S1), S2) S3) . Then the minimal equation of the matrix $\beta_o = \beta(\frac{p_o}{m})$ where $\vec{p} = (m', \vec{0})$ is :

$$\beta_o^n (\beta_o^2 - 1) = 0 \quad \text{III.1.13}$$

for some integer n .

Proof : We have shown that σ is the hyperboloid $p^2 = m'^2$. Therefore the spectrum of β_o is reduced to $\pm 1, 0$. Because of the covariance law one finds $R \beta_o R = -\beta_o$; thus, +1 and -1 have the same multiplicity.

On the other hand, if $T(p)$ is a covariant distribution solution of III.1.9 b), its support is contained in σ , and one can decompose T as a covariant linear combination of transverse derivatives of the invariant measure $d\Omega_m$:

$$T(p) = \sum_{\epsilon=\pm 1} \sum_{n=0}^{N_\epsilon} \left\{ \left(\frac{\partial}{\partial \sqrt{p^2}} \right)^n \left(\frac{\delta(p^0 - \epsilon \omega_p)}{2\omega_p} \right) \right\} (L[p] C_n^\epsilon L[p]^{-1}) \quad \text{III.1.14}$$

Putting $z = \pm \sqrt{p^2}$ if $p \in V^{\pm}$, equation III.1.9 b) becomes by a suitable redefinition of $C_n^{\pm}$:

$$\sum_{n=0}^{N_{\pm}} \delta^{(n)} \left(\frac{z}{m'} - \epsilon \right) (\beta_0 z - 1) C_n^{\epsilon} = 0 \quad \text{III.1.15}$$

which can be written in the form :

$$(\beta_0 - \epsilon) C_n^{\epsilon} = 0, (\beta_0 - \epsilon) C_{n-1}^{\epsilon} = \beta_0 C_n^{\epsilon}, \dots, (\beta_0 - \epsilon) C_0^{\epsilon} = \beta_0 C_1^{\epsilon} \quad \text{III.1.16}$$

A necessary and sufficient condition to have

$$C_1^{\epsilon} = C_2^{\epsilon} = \dots = C_n^{\epsilon} = 0 \quad \text{III.1.17}$$

is that the eigennilpotents corresponding to the eigenvalues ± 1 , be zero [12]. This proves the lemma.

If one modifies $\beta(p)$ in such a way that $m' = m$ one finds :

Corollary III.1.4 : The matrices $\beta(p)$ satisfy :

$$(\beta(p) - m)^{-1} = d(p) (p^2 - m^2)^{-1} \quad \forall p \in \mathbb{R} \quad \text{III.1.18}$$

where $d(p)$ is the Klein Gordon divisor 5,6,22 :

$$d(p) = (\beta(p) + m) \left[\frac{\beta(p)}{m} \right]^n - \left(\frac{p^2 - m^2}{m} \right) \sum_{k=0}^{n-1} \left(\frac{\beta(p)}{m} \right)^k \quad \text{III.1.19}$$

If $p^2 > 0$ the eigen-projectors of $\beta(p)$ corresponding to the eigenvalues $\pm \sqrt{p^2}$ are

$$\Pi_{\pm}(p) = \frac{1}{2} \left(\frac{\beta(p)}{\sqrt{p^2}} \right)^n \left(1 \pm \frac{\beta(p)}{\sqrt{p^2}} \right) \quad \text{III.1.20}$$

Proof : These results are standard consequences of the covariance and spectral theorems on finite dimensional matrices.

We are now able to characterize the free fields of interest, within the class we have considered.

Proposition III.1.5 :

Let ψ be an irreducible charged Wightman field, fulfilling equation III.1.11, and axioms S1, S2, S3. Then ψ is a free field with mass $m > 0$ and spin " S ", and the eigenprojector P_0 is given by :

$$P_0 = \alpha \eta \Pi_+(\vec{p}) \quad , \quad \alpha > 0, \quad \vec{p} = (m, \vec{0}) \quad \text{III.1.21}$$

Proof : If ψ is such a field one can easily check (cf. equation III.1.2) that

$$P_0(1 - \beta_0) = (\beta_0^* - 1)P_0 = 0 \quad \text{III.1.22}$$

Therefore ηP_0 commutes with β_0 and is annihilated to the right and the left by $\beta_0 - 1$. Since the eigennilpotent of β_0 for the eigenvalue $+1$ is zero, we get

$$\eta P_0 = \eta P_0 \Pi_+ = \Pi_+ \eta P_0 \quad \text{III.1.23}$$

where, for short, we have put $\Pi_+(\vec{p}) = \Pi_+$

On the other hand, since $\eta \Pi_+$ is hermitian, we get :

$$\text{Im}(\eta \Pi_+) = [\text{Ker}(\eta \Pi_+)]^\perp \quad \text{III.1.24}$$

Therefore the initial and final projectors [31] of $\eta \Pi_+$ are equal to the same projector P , which is of course $SU(2)$ -invariant and, by equation III.1.23

$$P_0 \preceq P \quad \text{III.1.25}$$

If $P_0 \neq P$, there exists P'_0 orthogonal to P_0 less than $P - P_0$ $SU(2)$ -invariant and irreducible such that III.1.22 holds for P'_0 . This would imply the existence of another solution for ψ , which is excluded by S3. Hence :

$$P_0 = P \quad \text{III.1.26}$$

Thus, by definition of P , and the Schur Lemma,

$$\eta \Pi_+ = \eta \Pi_+ P_0 = P_0 \eta \Pi_+ = P_0 \eta \Pi_+ P_0 = \alpha^2 P_0 \quad \text{III.1.27}$$

α is a real number since $\eta \Pi_+$ is hermitian, and α can be chosen positive by possibly changing the sign of η .

Definition III.1.6 :

$\mathcal{H}_\pm$ will denote the Hilbert space spanned by classes of functions on the positive mass-shell with values in E_N , and defined by the inner product

$$(\varphi|\varphi')_\pm = \int d\Omega_m(p) \langle \varphi(p), (\pm)^{2\Delta+n} \eta \Pi_\pm(p) \varphi'(p) \rangle \quad \text{III.1.28}$$

This definition is motivated by the form of the two point functions

$$L[p]^{*-1} \eta \Pi_+ L[p]^{-1} = \eta \Pi_+(p) \quad \text{a)}$$

$$\begin{aligned} L[p]^{*-1} R \eta \Pi_+ R^{-1} L[p]^{-1} &= (-)^{2\Delta} \eta R \Pi_+(p) R^{-1} \\ &= (-)^{2\Delta+n} \eta \Pi_-(p) \quad \text{b)} \end{aligned} \quad \text{III.1.29}$$

where use has been made of $R = L(1, -1) = L(-1) \eta R \eta^{-1}$ and Eq.(III.1.20).

III.2 Nice Interactions :

Let us examine the problem where ψ is coupled to an external field through the following interaction Lagrangean density :

$$\mathcal{L}_I(x; \Phi) = : \psi^*(x) \Phi(x) \psi(x) : \quad \text{III.2.1}$$

where $\Phi \in \mathcal{S}(\mathbb{R}^4) \otimes \mathcal{L}(E_N)$.

In the introduction we have defined "nice" interactions (cf., def. I.2) and we want to show that in this case, the problem can be reduced to the treatment of a scalar quantized field interacting with an external field as far as the existence of the Schwinger kernel is concerned. In order to do so we will use the equivalence theorems ([1] theorem II.4.1), classical theorems about the Klein-Gordon operators (see Appendix) and the techniques applied in section II to the case of weak external fields.

First of all, one has to exhibit some non trivial nice interactions

Proposition III.2.1 :

Let ψ be the Dirac field. Then any external field in $\mathcal{C}_0^\infty(\mathbb{R}^4) \otimes \mathcal{L}(E_N)$ is nice.

Proof : Take $\beta_\mu = \gamma_\mu$, and (cf. Eq. I.4) $B_R = B_L = \Phi$.

Then :

$$\begin{aligned} A_R &= -\Phi^2 - 2m\Phi - i(\gamma\partial\Phi - \Phi\gamma\partial) & \text{a)} \\ A_L &= -\Phi^2 - 2m\Phi + i(\gamma\partial\Phi - \Phi\gamma\partial) & \text{b)} \end{aligned} \quad \text{III.2.2}$$

since

$$(-i\gamma\partial + m + \Phi)(-i\gamma\partial - m - \Phi) = -(\square + m^2) + A_L \quad \text{a)} \quad \text{III.2.3}$$

$$(-i\gamma\partial - m - \Phi)(-i\gamma\partial + m + \Phi) = -(\square + m^2) + A_R \quad \text{b)}$$

and A_R, A_L are differential operators of degree one.

Proposition III.2.2 [24]

If one assumes

$$\beta(p)(\beta(p)^2 - p^2 \mathbb{1}) = 0 \quad \forall p \in \mathbb{R}^4 \quad \text{III.2.4}$$

then, the minimal coupling is a nice interaction

Proof : If $A_\mu \in \mathcal{C}_0^\infty(\mathbb{R}^4)$ then the minimal coupling is obtained by replacing $-i\partial_\mu$ by $\nabla_\mu = -i\partial_\mu - A_\mu$ in the field equations. Thus $\Phi = \beta_\mu A^\mu$.

Let us choose B_R and B_L in such a way that

$$d(-i\partial) + B_R = d(\nabla) = d(-i\partial) + B_L \quad \text{a) III.2.5}$$

Because of equation III.2.4, one finds :

$$d(\nabla) = m + \beta(\nabla) + \frac{\beta(\nabla)^2 - \nabla_\mu \nabla^\mu}{m} \quad \text{b) III.2.5}$$

Therefore

$$d(\nabla)(\beta(\nabla) - m) = \nabla_\mu \nabla^\mu - m^2 + \frac{\beta(\nabla)^3 - \nabla_\mu \nabla^\mu \beta(\nabla)}{m} \quad \text{III.2.6}$$

The last term in the right hand side can be written as :

$$-\frac{1}{m} (\beta_{\mu_1} \beta_{\mu_2} \beta_{\mu_3} - g_{\mu_1 \mu_2} \beta_{\mu_3}) (i\partial^{\mu_1} - A^{\mu_1}) (i\partial^{\mu_2} - A^{\mu_2}) (i\partial^{\mu_3} - A^{\mu_3}) \quad \text{III.2.7}$$

Equation III.2.4 tells us that the symmetric part of the tensor

$\beta_{\mu_1} \beta_{\mu_2} \beta_{\mu_3} - g_{\mu_1 \mu_2} \beta_{\mu_3}$ is zero, therefore, all symmetric terms in the expansion of $\nabla^{\mu_1} \nabla^{\mu_2} \nabla^{\mu_3}$ disappear. Using the commutation relation $[\partial^\mu, A^\nu] = (\partial^\mu A^\nu)$, one can see that

III.2.7 is a partial differential operator of degree less than one.

Remark III.2.3 :

One should first note that Prop. III.2.2 covers the Petiau Duffing-Kemmer theory [26]. Using general results about the matrices β_μ (see for instance [18] or [19]) one can construct all solutions β such that eq. III.2.4 holds. In particular, for any spin "s", it is possible to find such β 's. Unfortunately, it can be easily seen that if $s \geq 3/2$, the projector $\eta \pi_+$ (see proposition III.1.5) is not irreducible, under $SU(2)$ but reducible into a direct sum of an even number of $SU(2)$ -irreducible projectors. This fact leads to difficulties [25] in the particle interpretation of the quantized out field.

Proposition III.2.4:

The dipole coupling in the Petiau-Duffin-Kemmer equations for spin 1 fields is nice

Remark III.2.5 :

It has been remarked by Velo and Zwanziger that combining the minimal and dipole interactions the P.D.K. theory does not produce any acausality.

The nice character of these interactions confirms this result.

Proof : The P.D.K. field can be looked at as the 10-component field [26] $\psi = (V_\nu, G_{\mu\nu})$ which fulfills the system

$$\partial^\nu G_{\mu\nu} + m V_\mu = 0, \quad \partial_\mu V_\nu - \partial_\nu V_\mu + m G_{\mu\nu} = 0 \quad \text{III.2.8}$$

The dipole coupling is defined by :

$$\Phi.\psi = (\partial F^{\nu\rho} V_\rho, 0). \quad \text{III.2.9}$$

where $F^{\nu\rho}$ is some skew-symmetric tensor with $\mathcal{C}_0^\infty(\mathbb{R}^4)$ coefficients.

The choice $B_R = B_L = B$ with

$$B\psi = \left(\frac{i}{2m} F^{\rho\sigma} \partial_\nu G_{\rho\sigma}, 0 \right) \quad \text{III.2.10}$$

leads to first order partial differential operators for A_R, A_L .

III.3 The retarded kernel:

Proposition III.3.1 :

Let Φ be a nice interaction. Then $1 - \Phi S_r$ (resp. $1 - \Phi S_a$) is an isomorphism of $\mathcal{S}(\mathbb{R}^4) \otimes E_N$, and $\lambda \mapsto (1 - \lambda \Phi S_r)^{-1}$ (resp. $\lambda \mapsto (1 - \lambda \Phi S_a)^{-1}$) is an entire analytic function, in the simple convergence topology.

Proof : First of all, we recall that (see Lemma II.2.3, and Appendix) $1 - A_R(\lambda) \Delta_a$, $1 - A_L(\lambda) \Delta_a$ (where $A_{R,L}(\lambda)$ is computed by multiplying Φ by λ) are isomorphisms of $\mathcal{S}(\mathbb{R}^4) \otimes E_N$. On the other hand, by a classical theorem on distributions, $1 - \Phi S_r$ is a linear continuous map from $\mathcal{S}$ to $\mathcal{S}$.

If we prove that $1 - \Phi S_r$ is one-to-one then by the closed graph theorem ([27] Theor. 2.15) $1 - \Phi S_r$, is an isomorphisms.

Since Φ is nice one has $\forall f \in \mathcal{S}$:

$$\begin{aligned} (d(-i\partial) + B_L)(1 - \Phi S_r) f &= (d(-i\partial) + B_L) (\beta(-i\partial) - m - \Phi) S_r f \\ &= -(\square + m^2 - A_L) d(-i\partial) \Delta_r f = -(1 - A_L \Delta_r) d(-i\partial) f \end{aligned} \quad \text{III.3.1}$$

by repeated use of the associativity of convolutions [9]. Therefore, since $1 - A_L \Delta_r$ is an isomorphism of $\mathcal{T}$, $(1 - \Phi \mathcal{S}_r) f = 0$, $f \in \mathcal{T}$, implies

$$(d(-i\partial) f = 0, f \in \mathcal{T}) \iff f = 0 \quad \text{III.3.2}$$

because d is a polynomial

In much the same way, if $h \in \mathcal{T}$:

$$\begin{aligned} (1 - A_R \Delta_r) h &= (\square_{+m^2} - A_R) \Delta_r h = -(-i\beta_\mu \partial^\mu - \phi) / (d(-i\partial) + B_R) \Delta_r h \\ &= -(1 - \Phi \mathcal{S}_r) (-i\beta_\mu \partial^\mu - m) (d(-i\partial) + B_R) \Delta_r h. \\ &= -(1 - \Phi \mathcal{S}_r) [1 + (i\beta_\mu \partial^\mu + m) B_R \Delta_r] h \end{aligned} \quad \text{III.3.3}$$

Therefore, if $g \in \mathcal{T}$, the function:

$$f = -[1 + (i\beta_\mu \partial^\mu + m) B_R \Delta_r] (1 - A_R \Delta_r)^{-1} g \quad \text{III.3.4}$$

is a solution of

$$(1 - \Phi \mathcal{S}_r) f = g \quad \text{III.3.5}$$

Remark that here, the existence and associativity of the convolution is due to the fact that both $\mathcal{S}_r$ and Δ_r in III.3.3 have their supports in $\overline{V^+}$.

Finally the analyticity in λ of $(1 - A_R(\lambda) \Delta_r)^{-1}$, leads to the analyticity of $(1 - \lambda \Phi \mathcal{S}_r)^{-1}$.

Corollary III.3.2 :

The kernel $\mathcal{J}_r = (1 - \Phi \mathcal{S}_r) \Phi$ (resp. $\mathcal{J}_a = (1 - \Phi \mathcal{S}_a) \Phi$) is a continuous map from $\mathcal{O}_H = \mathcal{O}_H(\mathbb{R}^4) \otimes E_{\mathcal{N}}$ to $\mathcal{T}$. It is therefore a regular kernel (see [1] def. II.2.1)

Corollary III.3.3 :

If Φ is a nice interaction, $1 - \mathcal{S}_r \Phi$ (resp. $1 - \mathcal{S}_a \Phi$) is an isomorphism of $\mathcal{O}_H$, and $\lambda \rightarrow (1 - \lambda \mathcal{S}_r \Phi)$ is an entire analytic function, in the simple convergence topology.

Proof : Indeed :

$$(1 - S_r \Phi)^{-1} = 1 + S_r T_r$$

III.3.6

III.4 N-P property of the Schwinger kernel :

The main theorem is the following :

Theorem III.4.1 :

Let Φ be a hermitian nice interaction. Then for small enough real λ the kernel

$$K_{ex}(\lambda) = (1 - \lambda \Phi S_{ex})^{-1} \lambda \Phi \quad \text{III.4.1}$$

is N-P, if $S_{ex} \in \{S_r, S_a, S_F, S_{\bar{F}}\}$.

-Moreover, if β_μ are the Dirac matrices, then $\lambda \Rightarrow K_{ex}(\lambda)$ is analytic for $|\lambda| < \epsilon$ in the N-P norm topology.

To prove this theorem, we need some preliminaries.

Lemma II.4.2 :

Let Φ be a nice interaction, and $T_r(\lambda)$ be the kernel $(1 - \lambda \Phi S_{ex})^{-1} \lambda \Phi$. Then there exists $P \geq 0$ and $\forall \alpha \geq 0, \exists \epsilon > 0$ such that $\lambda \Rightarrow \tilde{T}_r(\lambda)$ (where $\sim$ denotes the Fourier transform) is analytic, in the topology of $N_{E_N}^2(m, \alpha, P)$ $\forall m \geq 0$ for $|\lambda| < \epsilon(\alpha)$.

Proof : With the help of equation III.3.4 one finds :

$$T_r(\lambda) = -\lambda (1 + (i\beta_\mu \partial_{x+m}) B_R(\lambda) \Delta_r) (1 - A_R(\lambda) \Delta_r)^{-1} \Phi \quad \text{III.4.2}$$

By theorem III.1.1, $(1 - A_R(\lambda) \Delta_r)^{-1} \Phi$ is analytic in $|\lambda| < \epsilon(\alpha)$ for the topology of $N_{E_N}^2(m, \alpha, 1/4)$ $\forall m \geq 0 \quad \forall \alpha \geq 0$

Using Lemma A.5.6 in [1], $-(\partial_\mu \beta_\mu)^{M+m} B_R(\lambda) \Delta_r$ maps linearly and continuously $N_{E_n}(m, \alpha, 1/4)$ into $N_{E_n}(m, \alpha, P)$ for some $P \geq 0$ which depends on the degree of the differential operator $B_R(\lambda)$. This is true $\forall m \geq 0 \forall \alpha \geq 0$, because B_R has $\mathcal{C}_0^\infty$ coefficients. The analyticity is also preserved because is at most a polynomial in λ .

Corollary III.4.3 :

If in Lemma III.4.2, β_μ are the Dirac matrices, then P can be chosen equal to $1/4$.

Proof : Indeed, in this case $B_R(\lambda) = \lambda \Phi$. P can be computed using Lemma A.5.6 of [1] which yields the value $1/4$.

In particular, in this case, if $J_R(\lambda)$ is the operator on $\mathcal{H}_+ \oplus \mathcal{H}_- = \mathcal{H}$ associated with $\bar{J}_r(\lambda)$ (see § II.2), then :

$$\lambda \rightarrow J_R(\lambda)$$

is analytic if $|\lambda| < \epsilon$ in the N-P norm topology (see Corollary II.1.3 or Appendix 5 of [1]).

Lemma III.4.4 :

Let Φ be a nice interaction. Then the components $J_{r+-}(\lambda)$ and $J_{r-+}(\lambda)$ are Hilbert-Schmidt operators, and analytic in λ for $|\lambda| < \epsilon(\alpha)$ for α large enough.

Proof : the proof is the same as in Lemma A.5.5 of [1]. We have only to recall that the inner product in $\mathcal{H}$ depends on the spin through the projector $\Pi_+(\rho)$ which is, on the mass shell, a polynomial (see §III.1).

Now, we can prove the rest of theorem III.4.1. Since $\bar{J}_r$ is a regular kernel (cor. III.3.2), J_{r++} (resp. J_{r--}) is densely defined on a domain $\mathcal{D}_+$ (resp. $\mathcal{D}_-$) such that $J_{\epsilon\epsilon} \mathcal{D}_\epsilon \subset \mathcal{D}_\epsilon$, $\epsilon = \pm 1$ (see [1], prop. II.2.5). On the other hand, the unitarity relations give us ([1] table I and § II.3) :

$$(\mathbb{1}_+ + \partial J_{r++}) (\mathbb{1}_+ - \partial J_{r++}^*) = \mathbb{1}_+ + J_{r+} (J_{r+})^* \quad (2s \text{ even}) \quad a)$$

III.4.3

$$(\mathbb{1}_+ + \partial J_{r++}) (\mathbb{1}_+ - \partial J_{r++}^*) = \mathbb{1}_+ - J_{r+} (J_{r+})^* \quad (2s \text{ odd}) \quad b)$$

Therefore, J_{r++} (resp. J_{r--}) can be continued as a bounded operator. Thus J_r is N-P.

In much the same way J_a is N-P. Since $\lambda \rightarrow J_{r+-}(\lambda)$ is analytic in the Hilbert-Schmidt norm, for $|\lambda| < \epsilon$, $1 + J_{r+-}(\lambda) J_{a-+}(\lambda)$ ($2S$ even) or $1 - J_{r+-}(\lambda) J_{r-+}(\lambda)$ ($2S$ odd) is invertible for $|\lambda| < \epsilon'$, ϵ' small enough. Therefore, the Feynman analogs $I = K_F$ and $\bar{I} = K_{\bar{F}}$ can be constructed by the equivalence theorem ([1] theorem II.4.1)

This remark achieves the proof.

Remark III.4.5 :

If the spin S is greater than one, the argument concerning the analyticity property does not go through because $1 + i J_R(\lambda)$ could be unbounded when λ is not real.

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APPENDIX

STUDY OF THE RETARDED KERNEL

A.1 The operator $(1 - A\Delta_r)$

Theorem A.1.1 :

Let $(a_\alpha)_{\alpha=0,1,2,3,4}$ be functions in $\mathcal{C}_0^\infty(\mathbb{R}^4) \otimes \mathcal{L}(E)$ where E is some finite dimensional space. Then, the operator $(1 - A\Delta_r)$, with

$$A(x, \partial) = \sum_{\alpha=0}^3 a_\alpha(x) \frac{\partial}{\partial x^\alpha} + a_4(x) \quad \text{A.1.1}$$

is an automorphism of $\mathcal{V}(\mathbb{R}^4) \otimes E$, and

$\lambda \rightarrow (1 - \lambda A\Delta_r)^{-1}$ is an entire analytic function, in the simple convergence topology.

In order to prove this theorem one introduces the following family of semi-norms : if $f \in \mathcal{V}(\mathbb{R}^4) \otimes E$ then

$$\|f\|_{m, t_0, t_1}^2 = \sum_{|\mu| \leq m} \int_{t_0}^{t_1} ds \int_{\mathbb{R}^3} d^3\vec{x} \|\partial^\mu f(s, \vec{x})\|_E^2 = \sum_{|\mu| \leq m} \|\partial^\mu f\|_{0, t_0, t_1}^2 \quad \text{a)}$$

$$p_{m, t_0, t_1}(A) = \sup_{\alpha} \sup_{\substack{s \in [t_0, t_1] \\ \vec{x} \in \mathbb{R}^3}} \sup_{|\mu| \leq m} \|\partial^\mu a_\alpha(s, \vec{x})\|_E \quad \text{b)} \quad \text{A.1.2}$$

$$\begin{aligned} \text{Here } \mu &= (\mu_0, \mu_1, \mu_2, \mu_3) \in \mathbb{N}^4 & \partial^\mu &= \prod_{\alpha=0}^3 (\partial_\alpha)^{\mu_\alpha} \\ |\mu| &= \sum_{\alpha=0}^3 \mu_\alpha \end{aligned} \quad \text{A.1.3}$$

Let $\mathcal{H}_{m,t_0,t_1}$ be the Hilbert space obtained by completion of $\mathcal{C}_0^\infty([t_0,t_1] \times \mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{m,t_0,t_1}$.

Lemma A.1.2 :

Let $(f_n)_{n \geq 0}$ be a converging sequence in the space $\bigcup_{t_0, t_1} \bigcap_{m \geq 0} \mathcal{H}_{m,t_0,t_1}$ with the topology defined by the family $\|\cdot\|_{m,t_0,t_1}$ (inductive limit topology with respect to t_0, t_1). If there exist $N \in \mathbb{N}$ and a compact subset K of $\mathbb{R}^3$ such that

$$\text{supp } f_n \subset K \quad \forall n \geq N \quad \text{A.1.4}$$

then f_n converges in $\mathcal{C}_0^\infty(\mathbb{R}^3) \otimes E$

This Lemma follows from the definition of convergence in $\mathcal{C}_0^\infty(\mathbb{R}^3) \otimes E$ ([3], [9]).

Lemma A.1.3 :

The following estimate holds :

$$\|A f\|_{m,t_0,t_1} \leq C_{p_{m,t_0,t_1}}(A) \|f\|_{m+1,t_0,t_1} \quad \text{A.1.5}$$

where the positive constant C depends only on m .

Lemma A.1.4 :

(A priori Bound [28, 29]) Let f be in $\mathcal{H}_{m,t_0,t_1}$. Then

$$\|\Delta_r f\|_{m+1,t_0,t_1} \leq \int_{t_0}^{t_1} ds \|f\|_{m,t_0,s} \quad \text{a)} \quad \text{A.1.6}$$

$$\|\Delta_a f\|_{m+1,t_0,t_1} \leq \int_{t_0}^{t_1} ds \|f\|_{m,s,t_1} \quad \text{b)}$$

Proof these inequalities are well-known if $f \in \mathcal{C}_0^\infty$ with support contained in $]t_0, t_1[\times \mathbb{R}^3$ [28]. By completion, they hold for any f in $\mathcal{H}_{m, t_0, t_1}$.

Lemma A.1.5 : Let

Let t_0, t_1 be such that :

$$\bigcup_{\alpha} \text{supp } a_{\alpha} \subset]t_0, t_1[\times \mathbb{R}^3 \quad \text{A.1.7}$$

Then, $\forall f \in \mathcal{H}_{m, s_0, s_1}$

$$\|(A\Delta_r)^n f\|_{m, s_0, s_1} \leq k^n p_{m, s, s'}(A)^n \frac{(t_1 - t_0)^n}{n!} \|f\|_{m, s, s'} \quad \text{A.1.8}$$

The last inequality is obtained by recursion on A.1.5 and A.1.6.

Theorem A.1.1 is then proved since $a_{\alpha} \in \mathcal{C}_0^\infty(\mathbb{R}^4) \otimes \mathcal{L}(E) \Rightarrow$

$$\text{supp } (A\Delta_r)^n f \subset \bigcup_{\alpha} \text{supp } a_{\alpha} \quad \forall n \geq 1$$

A.2 Properties of the classical S-operator (spin zero case).

We are now in the case $E = \mathbb{C}$, and $\mathcal{H}_{\pm}$ coincide with the Hilbert space $\mathcal{L}^2(\frac{1}{2}(m^2 + \vec{p}^2)^{-1/2} d^3\vec{p})$. Then we want to show the following theorem.

Theorem A.2.1:

Using the notations introduced in theorem A.1.1, the kernel $(\Delta_+ - \Delta_-)(1 - \lambda A\Delta_r)^{-1} A$ ($\lambda \in \mathbb{C}$) defines an analytic family of bounded operators on $\mathcal{H}_+ \oplus \mathcal{H}_-$.

Remark A.2.2 :

This bounded operator is nothing but $\mathcal{T}_{\mathcal{R}}$ defined in [1] § II.
(see [1] prop. II.5.1).

Let us introduce the following notations :

For a function f on $\mathbb{R}^4$ we define $\tilde{f}$ and $\hat{f}$ as follows :

$$\hat{f}(t, \vec{x}) = \int e^{i(p^0 t - \vec{p} \cdot \vec{x})} \tilde{f}(p, \vec{p}) d^4 p = \int e^{-i \vec{p} \cdot \vec{x}} \hat{f}(t, \vec{p}) d^3 \vec{p} \quad A.2.1$$

If $\varphi \in \mathcal{H}_{\pm}$

$$\hat{\varphi}(t, \vec{x}) = \int e^{i(\omega_p t - \vec{p} \cdot \vec{x})} \varphi(\vec{p}) \frac{d^3 \vec{p}}{2\omega_p} \quad \omega_p = (m^2 + \vec{p}^2)^{1/2} \quad A.2.2$$

The restrictions of $\tilde{f}$ to the mass shell will be :

$$\hat{f}_{(\vec{p})}^{(\pm)} = \tilde{f}(\pm \omega_p, \vec{p}) \quad A.2.3$$

Finally

$$q_{m, t_0, t_1}(a_\alpha) = \sup_{\alpha, |\alpha| \leq m} \sup_{t_0 < s < t_1} \int |\hat{\partial}_\alpha^0(s, \vec{p})| d^3 \vec{p} \quad A.2.4$$

Let K be the operator defined by :

$$\hat{K} \hat{f}(t, \vec{p}) = (m^2 + \vec{p}^2)^{1/4} \hat{f}(t, \vec{p}) \quad A.2.5$$

and

$$A' = K A K^{-1}$$

A.2.6

Lemma A.2.3

One has :

$$i) (\Delta_{\pm} f) = -i\pi \widehat{f^{(\pm)}}(s, \vec{x})$$

$$ii) \|K \widehat{f}\|_{0, t_0, t_1}^2 = (2\pi)^3 (t_1 - t_0) \|f\|_{\mathcal{H}_{\pm}}^2 \quad A.2.7$$

$$iii) \|K \Delta_{\pm} f\|_{0, t_0, t_1}^2 = (2\pi)^2 \pi^2 (t_1 - t_0) \|f^{(\pm)}\|_{\mathcal{H}_{\pm}}^2$$

This is merely the result of a tedious calculation.

Lemma A.2.4 :

If $f^{(\pm)}, \left(\frac{\partial f}{\partial p_{\alpha}}\right)^{(\pm)} \in \mathcal{H}_{\pm}$ for $\alpha = 0, 1, 2, 3$ then

$$\lim_{\substack{t_1 \rightarrow +\infty \\ t_0 \rightarrow -\infty}} \frac{(K \Delta_+ f | K \Delta_- f)_{0, t_0, t_1}}{t_1 - t_0} = 0 \quad A.2.8$$

Proof : if $(\cdot | \cdot)_{0, t_0, t_1}$ is the scalar product in $\mathcal{H}_{0, t_0, t_1}$ one has :

$$(K \Delta_+ f | K \Delta_- f)_{0, t_0, t_1} = c \int_{t_0}^{t_1} ds \int_{\mathbb{R}^3} e^{i\omega_p s} \frac{d^3 p}{2\omega_p} \overline{\widetilde{f}(\omega_p, \vec{p})} \widetilde{f}(-\omega_p, \vec{p}) = \int_{t_0}^{t_1} F(s) ds$$

By Lebergue's Lemma ([27], Theorem 7.5) $\lim_{s \rightarrow \infty} sF(s) = 0$, and the result follows.

Lemma A.2.5 :

The following estimate holds :

$$\|K^{-1} A K f\|_{0, t_0, t} \leq C q_{0, t_0, t}((k a_{\alpha})) \|f\|_{1, t_0, t} \quad A.2.9$$

In order to prove this Lemma, we have used the inequality :

$$\frac{(1+\vec{p}^2)}{(1+\vec{q}^2)} \leq 2 (1+(\vec{p}-\vec{q})^2) \quad \text{A.2.10}$$

and the properties of the Fourier transform (notice that the norm

$\|f\|_{m, t_0, t_1}$ is easily computable with the Parseval equality in terms of $\hat{f}$).

Lemma A.2.6 :

If $\bigcup_{\alpha} \text{supp } a_{\alpha} \subset]t_0, t_1[\times \mathbb{R}^3$, then :

$$\|\Delta_r (K A K^{-1}) f\|_{0, S_0, S_1} \leq H(A)^n \frac{(t_1 - t_0)^n}{n!} \|f\|_{0, S_0, S_1} \quad \text{A.2.11}$$

Proof : Let $A^\dagger$ be the adjoint of the operator A in $\mathcal{H}_{0, t_0, t_1}$.

$A^\dagger$ is associated with a first order differential operator. Then :

$$(\hat{h}_1 (\Delta_r K A K^{-1})^n f)_{0, S_0, S_1} = ((K^{-1} A^\dagger K \Delta_r)^n \hat{h}_1, f)_{0, S_0, S_1} \leq C H(A)^n \frac{(t_1 - t_0)^n}{n!} \|\hat{h}_1\|_{0, S_0, S_1} \|f\|_{0, S_0, S_1} \quad \text{A.2.12}$$

Proof of the theorem A.2.1 :

One can remark that, if $J_r = (1 - \lambda A \Delta_r)^{-1} A$

$$(\Delta_+ - \Delta_-) J_r f = -i\pi \left[\widehat{(J_r f)^{(+)}} - \widehat{(J_r f)^{(-)}} \right] \quad \text{A.2.13}$$

with the notations introduced in the beginning.

Let now $f \in \mathcal{C}^\infty$. Since $a_{\alpha} \in \mathcal{C}_0^\infty$, by Lemma A.2.4 :

$$\| (A \Delta_r)^n A f \rangle^{(+)} \|_{\mathcal{H}_+}^2 + \| (A \Delta_r)^n A f \rangle^{-} \|_{\mathcal{H}_-}^2 = C \lim_{\substack{S_1 \rightarrow \infty \\ S_0 \rightarrow -\infty}} \frac{\| K (\Delta_+ - \Delta_-) (A \Delta_r)^n A f \|_{0, S_0, S_1}^2}{S_1 - S_0} \quad \text{A.2.14}$$

Since $\Delta_+ - \Delta_- = \Delta_r - \Delta_a$, we have also :

$$\begin{aligned} \|K(\Delta_+ - \Delta_-)(A\Delta_r)^n A f\|_{0, S_0, S_1}^2 &\leq 2 \left(\|(\Delta_r A')^{n+1} K f\|_{0, S_0, S_1}^2 + \|\Delta_a A' (\Delta_r A')^n K f\|_{0, S_0, S_1}^2 \right) \\ &\leq H(A)^n \frac{(t_1 - t_0)^n}{n!} \|K f\|_{0, S_0, S_1}^2 \end{aligned} \quad A.2.15$$

Now let $\varphi_{\pm}$ be in $\mathcal{S}(\mathbb{R}^3)$. Then : by Lemma A.2.3

$$\begin{aligned} &\|((A\Delta_r)^n A \hat{\varphi}_{\pm})^{(+)}\|_{\mathcal{H}_+}^2 + \|((A\Delta_r)^n A \hat{\varphi}_{\pm})^{(-)}\|_{\mathcal{H}_-}^2 \\ &\leq C \lim_{\substack{S_1 \rightarrow \infty \\ S_0 \rightarrow -\infty}} H(A)^n \frac{(t_1 - t_0)^n}{n!} \frac{\|K \hat{\varphi}_{\pm}\|_{0, S_0, S_1}^2}{S_1 - S_0} \\ &\leq H'(A)^n \frac{(t_1 - t_0)^n}{n!} \|\varphi_{\pm}\|_{\mathcal{H}_{\pm}}^2 \end{aligned} \quad A.2.16$$

Therefore, the operators from $\mathcal{H}_{\epsilon}$ to $\mathcal{H}_{\epsilon'}$,

$$\varphi_{\epsilon} \rightarrow ((A\Delta_r)^n A \hat{\varphi}_{\epsilon})^{(\epsilon')} \quad A.2.17$$

are bounded on $\mathcal{S}(\mathbb{R}^3)$ which is dense in $\mathcal{H}_{\epsilon}$. They can therefore be continued as a bounded operator from $\mathcal{H}_{\epsilon} \rightarrow \mathcal{H}_{\epsilon'}$, and their norm are majorized by $C \frac{1}{n!}$ for some positive C . Therefore, the perturbation expansion for $(\Delta_+ - \Delta_-)(1 - A\Delta_r)^{-1}A$ converges in the norm topology in $\mathcal{L}(\mathcal{H}_+ \oplus \mathcal{H}_-)$.

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IRREDUCIBLE LORENTZ COVARIANT FREE FIELDS

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ABSTRACT. We classify free fields describing assemblies of irreducible systems of mass $m > 0$ and spin s , and transforming as :

$$U(a, A) \psi_\alpha(x) U(a, A)^{-1} = S_{\beta\alpha}(A^{-1}) \psi_\beta(Ax + Aa).$$

under the Poincaré group, where $S(A)$ is a given finite dimensional representation of $SL(2, \mathbb{C})$.

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I - Introduction.

This paper is devoted to the classification of all the Wightman free fields which describe assemblies of elementary systems of mass $m > 0$ and spin s .

This subject has a long history but somehow no complete solution can be found in any classical text book on quantum field theory (1,2).

Recently, an exhaustive construction of free fields has however been given by J.J Loeffel and J.P. Laederman [3] and by P. Renouard [4].

The present analysis provides some details which may still be of pedagogical interest and allows an easy extension to the case where an internal symmetry group is involved, which however will not be dealt with here.

The main result is described by the following theorem.

Let $\psi_\alpha(x)$ be a Wightman field with covariance law :

$$(I.1) \quad U(a, A) \psi_\alpha(x) U(a, A)^{-1} = \psi_\beta(Ax + a) S_{\beta\alpha}(A^{-1})$$

where $A \rightarrow S(A)$ is a finite dimensional representation of $SL(2, \mathbb{C})$ in an N -dimensional Hilbert space

Theorem I.1 : A necessary and sufficient condition for ψ to be a free field describing assemblies of elementary particles of mass $m > 0$ and spin s is the following :

i) There exists an orthogonal projector P_0 on a closed subspace of E_N , irreducible under a spin- s restriction of S to $SU(2)$.

ii) The two-points Wightman functions are :

$$\forall g \in \mathcal{L}_{E_N}(\mathbb{R}^4) \quad \forall f \in \mathcal{L}_{E_N}(\mathbb{R}^4).$$

$$(I.2) \quad a) \quad (\Omega, \psi(g) \psi(f)^* \Omega) = |\Lambda| \int_{H_m} d\Omega_m(p) \langle \tilde{f}(p), S(p)^* P_0 S(p) \tilde{g}(-p) \rangle$$

$$b) \quad (\Omega, \psi(f)^* \psi(g) \Omega) = |\Lambda| \int_{H_m} d\Omega_m(p) \langle \tilde{f}(p), S(p)^* P_0 R^{-1} S(p) \tilde{g}(p) \rangle$$

$$c) \quad (\Omega, \psi(f) \psi(g) \Omega) = |\Lambda| \int_{H_m} d\Omega_m(p) \langle \tilde{f}(p), S(p) P_0 R^{-1} S(p) \tilde{g}(p) \rangle$$

Here : $\mathcal{F}_E(\mathbb{R}^4) = \mathcal{F}(\mathbb{R}^4) \otimes E_N$ [5]

- $d\mu(p)$ is the Lorentz invariant measure $2d^3p (m^2 + p^2)^{-1/2}$ on the positive mass-shell

$$(I.3) \quad H_m = \{p \in \mathbb{R}^4; p^0 = \omega_p = (p^2 + m^2)^{1/2}\}$$

$\tilde{g}(p)$ is the Fourier transform of g .

$$\lambda \in \mathbb{C}$$

The basis in E_N is chosen in such a way that (see Appendix)

$$(1.4) \quad S(A)^* = S(A^*)$$

J is an antilinear involution such that :

$$(1.5) \quad JS(A)J^{-1} = S(A^{**})$$

$$(1.6) \quad \mathcal{C} = S(\mathcal{E})$$

- for $p \in H_m$, $[p]$ is a solution of

$$(1.7) \quad [p][p]^* = \frac{p}{m} : \frac{p^0 + \vec{p} \cdot \vec{\sigma}}{m}$$

where $\vec{\sigma}$ denotes the set of Pauli matrices.

In Section II, we recall the Wightman axioms and some definitions concerning free fields.

In Section III, we prove Theorem I.1, and we give in Section IV the Fock space representation of these fields.

II - Fields.

Let us recall the Wightman axiom [1,2] for fields.

WO) States : We need :

- a Hilbert space $\mathcal{H}$, with inner product $(\cdot | \cdot)$: the space of states.

- a Hilbert space E_N with inner product $\langle \cdot, \cdot \rangle$: the space of covariance ; with $\dim E_N = N$

- the space $\mathcal{S}$ of $\mathcal{C}^\infty$ fast decreasing function over $\mathbb{R}^4$ [5] with values in E_N . If $f \in \mathcal{S}$ we put

$$(II.1) \quad f(x) = (2\pi)^{-2} \int e^{-ipx} \tilde{f}(p) d^4p$$

W1) Fields : The field ψ maps $\mathcal{F}$ into linear operators on $\mathcal{H}$; $\psi(f)$ and $\psi(f)^*$ are defined on a dense subspace $\mathcal{D}$ of $\mathcal{H}$ independent of $f \in \mathcal{F}$; moreover, if Φ , and Ψ are in $\mathcal{D}$:

$$f \longrightarrow (\Phi, \psi(f) \Psi)$$

is a linear and continuous map . We require also

$$(II.2) \quad \psi(f) \mathcal{D} \subset \mathcal{D} \quad \psi(f)^* \mathcal{D} \subset \mathcal{D}$$

W2) Relativistic covariance :

A unitary and strongly continuous representation $U(a, A)$ of the restricted Poincaré group, is given in $\mathcal{H}$. A representation of the group $SL(2, \mathbb{C})$, $S(A)$ is given in E_N . The following conditions are satisfied :

$$(II.3) \quad \begin{aligned} & \text{i) if } f \in \mathcal{F} : \quad (a, A) f(x) = S(A) f(A^{-1}(x-a)) \\ & \text{ii) } U(a, A) \psi(f) U(a, A)^{-1} = \psi((a, A) f) \\ & \text{iii) } U(a, A) \mathcal{D} \subset \mathcal{D} \end{aligned}$$

W3) Spectral condition :

Let $(P_\mu)_{\mu=0,1,2,3}$ be the infinitesimal generators of the translation group $a \rightarrow U(a, 1)$. Then :

i) $\{p=0\}$ is an isolated eigenvalue of P . The corresponding eigenspace is one-dimensional and generated by Ω : $\Omega \in \mathcal{D}$ and $\|\Omega\| = 1$

ii) the rest of the spectrum of P is in V^+ .

W4) Locality :

Two subsets X and Y of $\mathbb{R}^4$ are said to be space-like separated (denoted by $X \sim Y$) if

$$\forall x \in X \quad \forall y \in Y \quad (x-y)^2 < 0$$

Then if $\text{supp}(f) \sim \text{supp}(g)$:

$$(II.4) \quad [\psi^\#(f), \psi^\#(g)]_\tau \Big|_D = \psi^\#(f) \psi^\#(g) - (-1)^\tau \psi^\#(g) \psi^\#(f) \Big|_D = 0$$

Here $\psi^\#(f) = \psi(f)$ or $\psi(f)^*$ and $\tau = 0$ or 1 .

W5) Completeness :

The vacuum is cyclic with respect to the algebra generated by $\{\psi^\#(f)\}_{f \in \mathcal{S}}$.

Definition II.1 : A free field of mass $m > 0$ is a Wightman field ψ such that

$$(II.5) \quad \psi((\square + m^2)f) = 0$$

Proposition II.2 : If ψ is a free field of mass $m > 0$, then, the spectrum of $\mathbb{P}_\mu$ is equal to

$$\{0\} \cup H_m \cup H_{>2m}$$

where

$$H_m = \{p \in \mathbb{R}^4; p^0 = (\vec{p}^2 + m^2)^{1/2}\}$$

$$H_{>2m} = \bigcup_{\mu \geq 2m} H_\mu$$

Proof : Because ψ is a solution of the Klein-Gordon equation, its support is contained in $H_m \cup (-H_m)$. Therefore, by the spectral condition :

$$\mathbb{P}_\mu \psi^\#(f_1) \dots \psi^\#(f_n) \Omega = \sum_{i=1}^n \psi^\#(f_1) \dots \psi^\#(\partial_{\mu_i} f_i) \dots \psi^\#(f_n) \Omega$$

is zero except if

$$\text{supp}(\tilde{f}_1 \otimes \dots \otimes \tilde{f}_n) \quad \text{intersects}$$

$$\mathcal{V}_n = \{(p_1, \dots, p_n) \in \mathbb{R}^{4n}; p_i \in H_m \cup (-H_m), p_i \in \bar{V}^+, p_1 + \dots + p_k \in \bar{V}^+ \forall k=1, \dots, n\}$$

Thus, $\mathcal{V}_n$ is equal to

$$\{(p_1, \dots, p_n) \in \mathbb{R}^{4n}; \quad p_1 + \dots + p_i \in \mathcal{V}^f \quad \forall i=1, \dots, n\} \quad \text{where} \quad \mathcal{V}^f = \{0\} \cup H_m \cup H_{>2m}$$

This proves the result.

Definition II.3 : Let ψ be a free field of mass m and let $\mathcal{H}_1$ be the eigenspace corresponding to $H_m = \{p^2 = m^2; p^0 > 0\}$. $\mathcal{H}_1$ will be called the one-particle space.

We say that the free field ψ of mass $m > 0$ is irreducible of spin s if $U(a, A) \upharpoonright_{\mathcal{H}_1}$ is an irreducible unitary representation of mass m and spin s .

III - Proof of theorem I.1 :

1°) The two-point Wightman functions are :

$$(III.1) \quad W_+(f, g) = (\Omega | \psi(g) \psi(f)^* \Omega); \quad W_-(f, g) = (\Omega | \psi(f) \psi(g)^* \Omega)$$

$$W_0(f, g) = (\Omega | \psi(f) \psi(g) \Omega)$$

By the nuclear theorem of L. Schwartz [5] and the Poincaré covariance, we have :

$$(III.2) \quad \begin{aligned} a) \quad W_{\pm}(f, g) &= \int \langle \tilde{f}(p), \tilde{W}_{\pm}(p) \tilde{g}(p) \rangle d^4 p \\ b) \quad W_0(f, g) &= \int \langle \mathcal{C}^{-1} \tilde{f}(p), \tilde{W}_0(p) \tilde{g}(p) \rangle d^4 p \end{aligned}$$

where

$$(III.3) \quad \begin{aligned} i) \quad \tilde{W}_{\pm} &\text{ are tempered distribution with values in } \mathcal{L}(E_N) \\ ii) \quad S(A)^{*^{-1}} \tilde{W}_{\pm}(p) S(A)^{-1} &= \tilde{W}_{\pm}(Ap) \\ iii) \quad S(A) \tilde{W}_0(p) S(A)^{-1} &= \tilde{W}_0(Ap) \\ iv) \quad \mathcal{J} &\text{ is an antiunitary involution on } E_N \text{ such that :} \\ &\mathcal{J} S(A) \mathcal{J}^{-1} = S(A^{*T}) \quad \mathcal{C} = S(i\sigma_2) \end{aligned}$$

We will prove iii) and iv) later.

2°) Let us first examine the case of $\mathcal{W}_+$.

With the Klein Gordon equation and the spectral condition W3 we obtain (see definition II.2) :

$$(III.4) \quad \text{supp } \tilde{\mathcal{W}}_+ \subset -H_m$$

Because the inner product in $\mathcal{H}$ is positive we have :

$$(III.5) \quad \mathcal{W}_+(f, f) \geq 0 \quad \forall f \in \mathcal{F}$$

By the Bochner-Schwartz theorem [5], $\tilde{\mathcal{W}}_+$ is a positive measure on $-H_m$. Using the covariance law (eq. III.3) we deduce that (see Appendix 1)

$$(III.6) \quad \tilde{\mathcal{W}}_+(-p) = S([p])^{*-1} P_+ S([p])^{-1} d\mu_m(p)$$

where

$$(III.7) \quad \begin{aligned} \text{i)} \quad & P_+ \geq 0 \quad \text{in } \mathcal{L}(E_N) \\ \text{ii)} \quad & S(u) P_+ S(u)^{-1} = P_+ \quad \forall u \in SU(2) \end{aligned}$$

3°) In much the same way, we can show the $\tilde{\mathcal{W}}_0(p)$ have their supports in H_m . By a theorem of Borchers and Greenberg [6] we deduce that ψ is a generalized free field [1,2]. In particular, the n -points truncated Wightman functions are zero except for $n = 2$. Moreover, $\mathcal{H}$ can be identified with the Fock space over $\mathcal{H}_1$. Let us remark that because of the Klein Gordon equation, the spectrum of $\mathbb{P}$ is as in II.6, and $\mathcal{H}_1$ is the Hilbert space generated by the vectors $\psi(f)\Omega$ and $\psi(f)^*\Omega$ with $f \in \mathcal{F}$.

Then the irreducibility hypothesis (Def. II.2 b), equation III.7, and the Schur lemma [7,8] imply :

$$(III.8) \quad P_+ = \lambda_+ P_0$$

with

$$(III.9) \quad \begin{aligned} \text{i)} \quad & \lambda_+ > 0 \\ \text{ii)} \quad & P_0 = P_0^* = P_0^2 \\ \text{iii)} \quad & U \rightarrow S(u)P_0 \text{ is an irreducible representation of } SU(2) \text{ of spin } s \text{ on } P_0(E_N) \text{ (see App. 2).} \end{aligned}$$

4°) In much the same way we can construct $\mathcal{W}_-$. Then we have :

$$(III.10) \quad \mathcal{W}_\pm(f, g) = \int \langle \tilde{f}(\bar{z}(p)), S(\bar{L}(p))^{*-1} P_\pm S(\bar{L}(p))^{-1} \tilde{g}(\bar{z}(p)) \rangle d\Omega_m(p)$$

$$\text{with} \quad P_+ = \lambda_+ P_0 \quad P_- = \lambda_- P'_0$$

where $\lambda_\pm, P'_0$ also fulfills conditions III.9.

We shall now use the axiom of locality W4), expressed as follows :

$$(III.11) \quad \mathcal{W}_+(f, g) - (-)^{\epsilon} \mathcal{W}_-(f, g) = 0$$

$$\text{if} \quad \text{supp}(f) \sim \text{supp } g$$

If we put :

$$(III.12) \quad G_\pm(p) = S(\bar{L}(p))^{*-1} P_\pm S(\bar{L}(p))^{-1}$$

by Loeffel's lemma (A.22 of App.2) $G_\pm(p)$ is the restriction on H_m of a polynomial on $\mathbb{R}^4$, and by the covariance law, we find :

$$a) \quad G(-p) = (-)^{2\epsilon} R G_+(p) R$$

$$(III.13) \quad \text{with}$$

$$b) \quad R = S(1, -1)$$

Equation III.11 can be written,

$$(III.14) \quad \begin{aligned} 0 &= G_+(\partial) \Delta_+ - (-)^{\epsilon} G_-(-\partial) \Delta_- \\ &= \frac{1}{2} \{ (G_+(\partial) + (-)^{\epsilon} G_-(-\partial)) (\Delta_+ - \Delta_-) + (G_+(\partial) - (-)^{\epsilon} G_-(-\partial)) (\Delta_+ + \Delta_-) \} \end{aligned}$$

with

$$(III.15) \quad \Delta_+(x) = \Delta_-(-x) = \int e^{i p x} d\Omega_m(p)$$

We know that $[1, 2] \quad \Delta_+ - \Delta_- = 0$ if $x^2 < 0$ and that $\Delta_\pm$ can be analytically continued into the extended tube $[1, 2]$ as an invariant function under the complex Lorentz group $L_+^{\mathbb{C}}$ (see App. A.1.1).

Therefore :

$$(III.16) \quad \left(G_+(i\partial) - (-)^{\tilde{\epsilon}} G_-(i\partial) \right) \Delta_+ = 0$$

in the analyticity domain. This condition implies :

$$(III.17) \quad G_+(p) = (-)^{\tilde{\epsilon}} G_-(-p) \Big|_{p^2=m^2}, p \in \mathbb{C}^4$$

In particular

$$(III.18) \quad \lambda_+ P_o = (-)^{\tilde{\epsilon}+2\delta} \lambda_- R P_o' R$$

By the properties of $\lambda_{\pm}$ and P_o, P_o' we find :

$$(III.19) \quad (-)^{\tilde{\epsilon}+2\delta} = 1 \quad \lambda_+ = \lambda_- = \lambda \quad P_o' = R P_o R$$

We recover in particular the connection between spin and statistics.

5°) We now concentrate on $\mathcal{W}_o$.

First of all it is easy to prove the existence of an antiunitarity involution $\mathcal{J}$ such that :

$$(III.20) \quad \mathcal{J} S(A, B) \mathcal{J} = S(A^{*\tau}, B^{*\tau})$$

Indeed, let $|k, m; \ell, n; \eta\rangle$ be an orthonormal basis which reduces S :

$$(III.21) \quad S(A, B) |k, m; \ell, n; \eta\rangle = \sum_{m', n'} D_{mm'}^{(k)}(A) D_{nn'}^{(\ell)}(B) |k, m'; \ell, n'; \eta\rangle$$

Then $\mathcal{J}$ is defined as follows (see also Appendix 1, eq. A1.2.5)

$$(III.22) \quad \text{a) } \mathcal{J} |k, m; \ell, n; \eta\rangle = |k, m; \ell, n; \eta\rangle$$

and

$$\text{b) } \mathcal{J}(\lambda x + \mu y) = \bar{\lambda} \mathcal{J}(x) + \bar{\mu} \mathcal{J}(y)$$

$$\forall \lambda, \mu \in \mathbb{C} \quad \forall x, y \in E_N$$

$$\text{c) We can also choose } \mathcal{J} \text{ such that } \mathcal{J} P_o \mathcal{J}^{-1} = P_o$$

Let us recall that $\mathcal{C}$ satisfies :

$$\mathcal{C} = S(i\sigma_2)$$

$$\text{a) } \mathcal{C}^* = \mathcal{C}^{-1} = S(-1) \mathcal{C}$$

$$(III.23) \quad \text{b) } [\mathcal{C}, P_o] = 0 = [\mathcal{C}, R]$$

$$\text{c) } S(A^{*\tau}) = \mathcal{C} S(A^{*-1}) \mathcal{C}^{-1}$$

We have also to note that, the scalar product $\langle, \rangle$ in E_N , is chosen in such a way that :

$$(III.24) \quad S(A)^* = S(A^*)$$

6°) As for the case of $\mathcal{W}_\pm$, using the covariance, the spectral condition, and the properties of $\mathcal{C}$, we can write :

$$(III.25) \quad \mathcal{W}_0(f, g) = \int \langle \mathcal{C}^{-1} J \tilde{F}(-p), S(\tilde{p}) \Gamma_0 S(\tilde{p})^{-1} \tilde{g}(p) \rangle d\mu_m(p)$$

with

$$(III.26) \quad [\Gamma_0, S(u)] = 0 \quad \forall u \in SU(2)$$

The one particle space $\mathcal{H}_1$ is generated by $\psi(f)\Omega$ as well as $\psi^*(f)\Omega$, for $f \in \mathcal{F}$, and $U(q, A)|_{\mathcal{H}_1}$ is irreducible. Therefore, we can deduce :

$$(III.27) \quad \Gamma_0 P'_0 = \Gamma_0 = P_0 \Gamma_0 = P_0 \Gamma_0 P'_0$$

Hence $(\Gamma_0 R) = P_0 (\Gamma_0 R) P_0$
Since P_0 is an irreducible projector under $SU(2)$, there exists in the Schur Lemma [7,8], a number λ' such that :

$$(III.28) \quad \Gamma_0 = \lambda' P_0 R$$

Thus, as one can verify, the locality axiom is automatically satisfied.

7°) Now, we know that $\mathcal{H}_1$ is generated by the vectors $\psi(f)\Omega$ as well as $\psi(f)^*\Omega$. In fact we have :

Lemma III.1 : $\mathcal{H}_1$ is the subspace of $\mathcal{H}$ generated by $(\psi(f)^*\Omega)_{f \in \mathcal{F}}$.

Proof : Indeed, let u be a mapping such that :

$$(III.29) \quad \begin{aligned} \text{i)} \quad & u : E_N \rightarrow E_{2N+1} \\ \text{ii)} \quad & u S(u) = \mathcal{D}^s(u) u \quad \forall u \in SU(2) \\ \text{iii)} \quad & uu^* = \mathbb{1}_{E_{2N+1}} \quad u^*u = P_0 \end{aligned}$$

where E_{2n+1} is the Hilbert space in which the irreducible representation $\mathcal{D}^{(s)}$ of $SU(2)$ is defined (see Appendix A.1.2). Such a mapping exists, and is defined up-to a phase, because of the irreducibility of P_0 under $SU(2)$ and the Schur Lemma [7,8].

Therefore, the mapping

$$(III.30) \quad \alpha: \psi(f)^* \Omega \longrightarrow u S([p])^{-1} \tilde{f}(-p)$$

can be continued as an isomorphism between the space $\mathcal{H}_{1,+}$ generated by $(\psi(f)^* \Omega)_{f \in \mathcal{Y}}$ and the space $\mathcal{H}_{m,S}$ of classes of functions γ from $SL(2, \mathbb{C})$ into E_{2n+1} such that :

$$(III.31) \quad \begin{aligned} \text{i)} \quad & \gamma(AU) = \mathcal{D}^{(s)}(U)^{-1} \gamma(A) \quad \forall A \in SL(2, \mathbb{C}), \forall U \in SU(2) \\ \text{ii)} \quad & \|\gamma\|_{m,S}^2 = \int \langle \gamma([p]), \gamma[p] \rangle d\Omega_m(p) < +\infty \end{aligned}$$

As is well known, [11,15] the representation $U_1(a, A)$:

$$(III.32) \quad [U_1(a, A) \gamma]([p]) = e^{i p \cdot a} \mathcal{D}^{(s)}(A) \gamma(A^{-1}[p]) \quad (a, A) \in \bar{\mathcal{G}}_+^{\uparrow}$$

is unitary and irreducible in $\mathcal{H}_{m,S}$ and we can see that :

$$(III.33) \quad \alpha \circ U(a, A) = U_1(a, A) \circ \alpha$$

Therefore, $\mathcal{H}_{1,+}$ is an irreducible closed subspace of $\mathcal{H}_1$, and because our field is irreducible we have

$$(III.34) \quad \mathcal{H}_1 = \mathcal{H}_{1,+}$$

Corollary III.2 : we have

$$(III.35) \quad |\lambda'| = \lambda = \lambda_{\pm}$$

Indeed, by Lemma III.1, we deduce :

$$(III.36) \quad \forall \epsilon > 0, \forall f \in \mathcal{Y}, \exists g \in \mathcal{Y} \quad \|\psi(f)\Omega - \psi(g)^*\Omega\| < \epsilon$$

But we have :

$$(III.37) \quad \|\psi(f)\Omega - \psi(g)^*\Omega\|^2 = W_+(g,g) + W_-(f,f) - 2\operatorname{Re} W_0(g,f)$$

Using the Schwarz inequality, it is easy to see that III.36 implies $|\lambda| = \lambda$.

Therefore, theorem I.1 is proved.

IV - Free fields in Fock space :

We have defined in Section III the Hilbert space $\mathcal{H}_{m,s}$ (eq. III.31/32) in which the irreducible representation U_1 of mass $m > 0$ and spin s of $\overline{\mathcal{B}}_+^{\uparrow}$ is defined.

Let $\mathcal{F}_{m,s}^{(\varepsilon)}$ be the Fock space of bosons (resp. fermions) constructed on $\mathcal{H}_{m,s}$ if $\varepsilon=0$ (resp. if $\varepsilon=1$). If $\gamma \in \mathcal{H}_{m,s}$, $a(\gamma)$ denotes the annihilation operator :

$$(IV.1) \quad \begin{aligned} \text{i)} \quad & a(\gamma) = \int d\mu_m(p) a([p]) \gamma([p]) \\ \text{ii)} \quad & a(\gamma)\Omega = 0 \quad \text{if } \Omega \text{ is the } \mathcal{F}_{m,s}^{(\varepsilon)}\text{-vacuum} \\ \text{iii)} \quad & \{a(\gamma), a(\gamma')^*\}_{\varepsilon} = (\gamma'|\gamma)_{m,s} \end{aligned}$$

$a^*(\gamma)$ denotes the creation operators :

$$(IV.2) \quad a^*(\gamma) = a(J_s \gamma)^* = \int d\mu_m(p) a^*([p]) \gamma([p])$$

where J_s is the antiunitary involution into E_{2s+1} such that

$$(IV.3) \quad J_s \mathcal{D}^{(s)}(A) J_s^{-1} = \mathcal{D}^{(s)}(A^{*\pi}) \quad \forall A \in SL(2, \mathbb{C})$$

(see III.20,21,22).

Then we have the following theorem :

Theorem IV.1 : Let ψ be a Wightman irreducible free field of mass $m > 0$ and spin s , satisfying I.1. Then there exists a mapping U such that :

$$\begin{aligned}
 & \text{i)} \quad u: E_N \rightarrow E_{2d+1} \\
 & \text{ii)} \quad uu^* = \mathbb{1}_{E_{2d+1}} \quad u^*u = P_0 \\
 & \text{iii)} \quad uS(V) = \mathcal{D}^{(s)}(V)u \quad \forall V \in SU(2) \\
 & \text{iv)} \quad \mathcal{I}_S u = u\mathcal{J}
 \end{aligned}
 \tag{IV.6}$$

and such that ψ can be realized as follows on $\mathcal{F}_{m,S}^{(\varepsilon)}$, if $(-)^{\varepsilon} = (-)^{2s}$

$$\psi(f) = \lambda^{1/2} \int d\Omega_m(p) \left\{ a(p) u S(p)^{-1} \tilde{f}(p) + a^*(p) v S(p)^{-1} \tilde{f}(p) \right\}
 \tag{IV.7}$$

with $f \in \mathcal{Y}$ and

$$v = u R \mathcal{C} \quad \mathcal{C} = S(i\sigma_2) \quad R = S(\mathbb{1}, -\mathbb{1})
 \tag{IV.8}$$

Proof : we know that u exists (eq. III.29), and it is possible to choose $\mathcal{J}$ such that $\mathcal{I}_S u = u\mathcal{J}$.

Therefore, it is sufficient to compute the two-point Wightman functions to see that the conclusion of Theorem IV.1 holds.

Remark IV.2 : It is clear that u and v play the role of the Dirac spinors [9,10,11].

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APPENDIX 1 : Finite Irreducible Representation
of the Lorentz Group.

A.1.1. - SL(2, C) versus $\mathcal{L}_+^\uparrow$:

If $p = (p_0, \vec{p}) \in \mathbb{R}^4$ we put :

$$(A.1.1.1.) \quad \tilde{p} = p^0 - \vec{p} \cdot \vec{\sigma} \quad p = p^0 + \vec{p} \cdot \vec{\sigma}$$

where $(\sigma_i)_{i=1,2,3}$ are the Pauli matrices :

$$(A.1.1.2) \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

we have :

$$(A.1.1.3) \quad \begin{aligned} \text{a)} \quad & \tilde{p} \tilde{p} = p^0 - \vec{p}^2 = p^2 \\ \text{b)} \quad & \det \tilde{p} = \det p = p^2 \\ \text{c)} \quad & \frac{1}{2} \operatorname{tr}(\tilde{p}) = p^0 = \frac{1}{2} \operatorname{tr}(p) \\ \text{d)} \quad & -\frac{1}{2} \operatorname{tr}(\sigma_i \tilde{p}) = p_i = \frac{1}{2} \operatorname{tr}(\sigma_i p) \end{aligned}$$

If $A, B \in SL(2, \mathbb{C})$ we put :

$$(A.1.1.4) \quad \underbrace{(A, B)}_p = A p B^{-1}$$

Then, $(\underbrace{(A, B)}_p)^2 = p^2$ and $p \rightarrow \underbrace{(A, B)}_p$ defines a complex Lorentz transformation. It is a real transformation if and only if $B = A^{*-1}$. Then $p \rightarrow \underbrace{(A, A^{*-1})}_p$ belongs to the restricted Lorentz group.

We will denote :

$$(A.1.15) \quad SL(2, \mathbb{C}) = \overline{\mathcal{L}_+^\uparrow} \quad SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) = \overline{\mathcal{L}_+^{\mathbb{C}}}$$

It can be seen that $\overline{\mathcal{L}_+^\uparrow}$ and $\overline{\mathcal{L}_+^{\mathbb{C}}}$ are the covering groups of $\mathcal{L}_+^\uparrow$ and $\mathcal{L}_+^{\mathbb{C}}$ respectively.

Correspondingly, $SU(2)$ (resp. $SU(2) \times SU(2)$) is the covering group of $SO(3)$ (resp. $S O(4)$). The transformation $p \rightarrow \underbrace{(1, -1)}_p$ is the total inversion :

$$(A.1.1.6) \quad \underbrace{(1, -1)}_p = -p$$

Finally if we put $C = i\sigma_2$ we have :

$$(A.1.1.7) \quad C^* = C^{-1} = -C \quad C^T = C$$

$$CAC^{-1} A^T = \det A \quad \forall A \in \mathcal{L}(\mathbb{C}^2)$$

A.1.2. - The irreducible representation $\mathcal{D}^j$:

It can be shown [12, 13, 14, 15, 16] that any irreducible representation of $SU(2)$ is equivalent to one of the representation defined as follows [12] :

- Let E_{2j+1} be the Hilbert space of polynomial $P(\zeta)$, $\zeta \in \mathbb{C}^2$ which are homogeneous of degree $2j$. The inner product is given by :

$$(A.1.2.1) \quad (P, Q)_j = \int P(\zeta)^* Q(\zeta) e^{-\langle \zeta, \zeta \rangle} \frac{d^4 \zeta}{\pi^2}$$

with $d^4 \zeta = d^2 \xi d^2 \eta$ if $\xi + i\eta = \zeta$ $\xi, \eta \in \mathbb{R}^2$

An orthonormal basis is given by :

$$(A.1.2.2) \quad P_{j,m}(\zeta) = \frac{\zeta_1^{j+m} \zeta_2^{j-m}}{[(j+m)! (j-m)!]^{1/2}} \quad \begin{array}{l} m = -j, -j+1, \dots, j-1, j \\ \zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2 \end{array}$$

$\mathcal{D}^j$ is then defined by :

$$(A.1.2.3) \quad \mathcal{D}^j(A) P(\zeta) = P(A^{-1} \zeta) \quad A \in \mathcal{L}(\mathbb{C}^2)$$

Then $\mathcal{D}^j|_{SU(2)}$ is unitary and we have :

$$(A.1.2.4) \quad \mathcal{D}^j_{m,m'}(A) = (P_{j,m} | \mathcal{D}^j(A) P_{j,m'})_j =$$

$$\sum_{\substack{p+q=j-m, \\ 0 \leq p \leq j+m \\ 0 \leq q \leq j-m}} \frac{((j+m')! (j-m')! (j+m)! (j-m)!)^{1/2}}{(j+m-p)! p! (j-m-q)! q!} a^{j+m-p} b^p c^{j-m-q} d^q$$

if
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\begin{aligned} \text{a)} \quad & \mathcal{D}_{mm'}^j(A^*) = \overline{\mathcal{D}_{m'm}^j(A)} \\ \text{(A.1.2.5) b)} \quad & \mathcal{D}_{mm'}^j(A^{-1}) = \mathcal{D}_{m'm}^j(A) \\ \text{c)} \quad & \mathcal{D}_{mm'}^j(1) = \delta_{mm'} \end{aligned}$$

And $\mathcal{D}^j$ is a polynomial with respect to A .

Let $d\mu$ be the normalized Haar measure of $SU(2)$. Then :

$$\text{(A.1.2.6)} \quad \int_{u \in SU(2)} \overline{\mathcal{D}_{mm'}^j(u)} \mathcal{D}_{nn'}^{j'}(u) d\mu(u) = (2j+1) \delta_{jj'} \delta_{mm'} \delta_{nn'}$$

A.1.3. - Finite dimensional representation of $\overline{\mathcal{L}_+^{\mathcal{Q}}}$:

We know that $A \rightarrow \mathcal{D}^j(A)$ and $A \rightarrow \mathcal{D}^j(A^{*-1})$ are not equivalent.

It can be shown [14] that all finite dimensional irreducible representations of $SL(2, \mathbb{C})$ are of the form :

$$\text{(A.1.3.1)} \quad A \rightarrow \mathcal{D}^j(A) \otimes \mathcal{D}^k(A^{*-1}) \quad 2k \in \mathbb{N}, 2l \in \mathbb{N}$$

Then these representations can be analytically continued to $\overline{\mathcal{L}_+^{\mathcal{Q}}}$ by :

$$\text{(A.1.3.2)} \quad (A, B) \rightarrow \mathcal{D}^j(A) \otimes \mathcal{D}^k(B)$$

Now let E_N be a finite dimensional Hilbert space : $\dim E_N = N$

Let S be a representation of $\overline{\mathcal{L}_+^{\mathcal{Q}}}$ in E_N . It is completely reducible and therefore can be analytically continued to $\overline{\mathcal{L}_+^{\mathcal{Q}}}$ according to A.1.3.2.

In particular, S is characterized, as analytic function by its restriction on $\overline{SO(4)} = \overline{SU(2) \times SU(2)}$.

Therefore, the decomposition of S can be performed using its restriction on $SO(4)$:

$$\text{(A.1.3.3)} \quad S(u, v) = \sum_{(k, l) \in I} \sum_{\substack{m, m' \\ n, n'}} \mathcal{D}_{mm'}^k(u) \mathcal{D}_{nn'}^l(v) P_{(m, m', n, n')}^{(k, l)}$$

with

$$\text{(A.1.3.4)} \quad P_{(mm')/(nn')}^{(k, l)} = (2k+1)^{-1} (2l+1)^{-1} \int d\mu(u) d\mu(v) \overline{\mathcal{D}_{mm'}^k(u)} \overline{\mathcal{D}_{nn'}^l(v)} S(u, v)$$

Hence :

$$\begin{aligned}
 \text{a)} \quad S(A, B) &= \sum_{(kl) \in I} \sum_{mm', nn'} \mathcal{D}_{mm'}^k(A) \mathcal{D}_{nn'}^l(B) P_{(mm')(nn')}^{kl} \\
 \text{b)} \quad P_{(mm')(nn')}^{kl} P_{(rs')(st')}^{ij} &= \delta_{ik} \delta_{jl} \delta_{m'r} \delta_{n's} P_{m'r', n's'}^{kl} \\
 \text{c)} \quad P^{kl} &= \sum_{m, m'} P_{mm'}^{kl} \quad \text{are projectors.} \\
 \text{d)} \quad P^{kl} P^{k'l'} &= \delta_{kk'} \delta_{ll'} P^{kl}
 \end{aligned}$$

Then the set I of indices is finite because

$$\text{e)} \quad \sum_{(kl) \in I} P^{(kl)} = \mathbb{1}_{E_N}$$

One can choose the inner product in E_N , such that $S|_{\overline{SO(4)}}$ is unitary because $\overline{SO(4)}$ is a compact group. Therefore :

$$\begin{aligned}
 \text{a)} \quad P_{mm', nn'}^{kl} &= P_{m'm, n'n}^{kl} \\
 \text{b)} \quad P^{kl} &= (P^{kl})^* \\
 \text{c)} \quad S(A^*, B^*) &= S(A, B)^*
 \end{aligned}$$

APPENDIX 2 : Covariant Functions on the Mass-Shell.

A.2.1. - Eigen projectors of SU(2) :

Let E_N be a finite dimensional Hilbert space with $N = \dim E_N$. Let S be a representation of $\overline{L_+^{\uparrow}}$ defined in E_N . Then the restriction of S to $SU(2)$ can be decomposed into irreducible representations :

$$(A.2.1.1) \quad S(u) = \sum_{j \in J} \sum_{m, m'} \mathcal{D}_{mm'}^j(u) \underline{P}_{mm'}^j$$

with

$$(A.2.1.2) \quad \underline{P}_{mm'}^j = \frac{1}{(2j+1)} \int d\mu \overline{\mathcal{D}_{mm'}^j(u)} S(u).$$

Then

$$(A.2.1.3) \quad \sum_{m, m'} \underline{P}_{mm'}^j = \underline{P}^j$$

is an orthonormal projector, and $\underline{P}^j S(u) \underline{P}^j$ is a direct sum of irreducible representation equivalents to $\mathcal{D}^j$. One has in particular :

$$(A.2.1.4) \quad \begin{aligned} \text{a)} \quad & S(u) \underline{P}^j = \underline{P}^j S(u) \quad \forall u \in S(U(2)) \\ \text{b)} \quad & \frac{1}{2j+1} \text{tr}(\underline{P}^j) \in \mathbb{N} \\ \text{c)} \quad & S(\Lambda \mathbb{1}) \underline{P}^j = \Lambda^2 \underline{P}^j = \underline{P}^j S(\Lambda \mathbb{1}). \end{aligned}$$

More generally, one has :

$$(A.2.1.5) \quad \forall A \in \overline{L_+^{\uparrow}} \quad \underline{P}^k S(A) \underline{P}^l = 0 \quad \text{if} \quad (-)^{2k} \neq (-)^{2l}$$

$$\text{Indeed, we have} \quad S(-\mathbb{1}) \underline{P}^l = (-)^{2l} \underline{P}^l = \underline{P}^l S(-\mathbb{1}) \quad \forall l \in J$$

$$\text{Thus :} \quad \underline{P}^k S(A) \underline{P}^l = (-)^{2k} \underline{P}^k S(-A) \underline{P}^l = (-)^{2l} \underline{P}^k S(-A) \underline{P}^l$$

By analytic continuation to $\overline{L_+^{\uparrow}}$, one finds :

$$(A.2.1.6) \quad \forall (A, B) \in \overline{L_+^{\uparrow}} \quad \underline{P}^k S(A, B) \underline{P}^l = 0 \quad \text{if} \quad (-)^{2k} \neq (-)^{2l}$$

In particular, let $\underline{P}_0$ be a projector such that :

$$(A.2.1.7) \quad S(u) \underline{P}_0 = \underline{P}_0 S(u)$$

and $U \rightarrow S(U)P_o$ equivalent to $\mathcal{D}^j$

Then, we call P_o an irreducible projector of spin j .

Therefore, the subspace E_{N_o} generated by P_o and $S(A, B)$ is such that :

$$(A.2.1.8) \quad S(-1) \big|_{E_{N_o}} = (-)^{2j} 1_{E_{N_o}}.$$

A.2.2. - Covariant functions of the mass-shell :

Let H_m be the positive mass-shell :

$$H_m = \{ p \in \mathbb{R}^4; \quad p^0 = \omega_p = (m^2 + \vec{p}^2)^{1/2} \}$$

$$(A.2.2.1) \quad H_m^{\mathbb{C}} = \{ p \in \mathbb{C}^4; \quad p^0^2 - \vec{p}^2 = m^2 \}$$

Definition : A covariant function on H_m with values in E_N is a function

$$a) \quad \varphi : H_m \rightarrow E_N$$

(A.2.2.2) with

$$b) \quad \varphi(Ap) = S(A) \varphi(p) \quad \forall A \in \overline{\mathcal{L}}_+^{\uparrow} \quad \forall p \in H_m$$

Lemma (Loeffel, [3]). Any covariant function on H_m is the restriction to H_m of a polynomial in $\mathbb{C}^4$. In particular, it can be continued to a covariant polynomial on $H_m^{\mathbb{C}}$.

Proof : Let us put : $\vec{p} = (m, \vec{0})$

Then :

$$(A.2.2.3) \quad \varphi(\beta) = S(U) \varphi(\vec{p}) \quad \text{if} \quad U \in SU(2) = \overline{SO(3)}$$

By analytic continuation :

$$(A.2.2.4) \quad \varphi(\vec{p}) = S(A, A) \varphi(\vec{p}) \quad \forall A \in \overline{\mathcal{L}}_+^{\uparrow} = SL(2, \mathbb{C})$$

Then

$$\varphi(p) = S([p]) \varphi(\vec{p}) = S(\frac{p}{m}, 1) S([p]^*, [p]^*) \varphi(\vec{p})$$

$$(A.2.2.5) \quad = S(\frac{p}{m}, 1) \varphi(\vec{p}) \quad \forall p \in H_m$$

where $[p]$ is defined, (up to an element of $SU(2)$), by

$$(A.2.2.6) \quad [p][p]^* = \frac{p}{m} \quad \Rightarrow \quad p = [p] \frac{p}{m}$$

Because $S(A, B)$ is a polynomial with respect to (A, B) one finds easily the result of the Lemma.

In particular, we have :

$$(A.2.2.7) \quad \varphi(-p) = S\left(-\frac{p}{m}, 1\right) \varphi(\tilde{p}) = S(1, 1) \varphi(\tilde{p})$$

Applications : $\varphi_{i,2} : H_m \rightarrow \mathcal{L}(E_N)$

$$(A.2.2.8) \quad \varphi_1(p) = S(A) \varphi_1(p) S(A)^{-1} \quad \varphi_2(p) = S(A)^{*^{-1}} \varphi_2(p) S(A)^{-1}$$

if $P_i = \varphi_i(\tilde{p})$ we have :

$$(A.2.2.9) \quad \begin{aligned} \text{a)} \quad \varphi_1(p) &= S([p]) P_1 S([p])^{-1} = S\left(\frac{p}{m}, 1\right) P_1 S\left(\frac{\tilde{p}}{m}, 1\right) \\ \text{b)} \quad \varphi_2(p) &= S([p])^{*^{-1}} P_2 S([p])^{-1} = S\left(1, \frac{p}{m}\right) P_2 S\left(\frac{\tilde{p}}{m}, 1\right) \end{aligned}$$

And :

$$(A.2.2.10) \quad \begin{aligned} \text{a)} \quad \varphi_1(-p) &= R \varphi_1(p) R \quad \varphi_2(-p) = S(-1) R \varphi_2(p) R \\ \text{b)} \quad R &= R^{-1} = R^* = S(1, -1) \end{aligned}$$

with

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